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ON SHEAR REINFORCEMENT DESIGN OF STRUCTURAL CONCRETE BEAMS ON THE BASIS OF THEORY OF PLASTICITY

Bjarne Christian Jensen[1], **Andrzej Lapko**[2]
[1]*University of Southern Denmark, Campusvej 55, DK-5230 Odense M, Denmark*
[2]*Bialystok Technical University, ul. Wiejska 45E, 15-351 Bialystok, Poland*
E-mails: [1]*bcj@ib.sdu.dk;* [2]*lapko@pb.edu.pl*

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Abstract. Modern design of reinforced concrete structural members for shear is based on the theory of plasticity. This paper is written to contribute to the understanding of the inclination of the concrete strut in the inclined strut model for design of shear reinforcement in beams, which among others are used in Eurocode 2. The problem of inclination of the compression strut in truss model is analysed depending on shear reinforcement ratio and effectiveness ratio of concrete strength for compression. Also the understanding of necessary ductility in steel reinforcing bars is discussed in the paper and especially the needs of tests on translation capacity of the shear failure are here analysed. To explain these problems the paper gives a short introduction to the theory of plasticity of reinforced concrete in shear and the background for the equations, which are used in shear design according to Eurocode 2.

Keywords: RC beams, shear design, plasticity of concrete, stringer method, inclined crack width.

1. Introduction

Recent research concerning shear capacity of reinforced concrete members are concentrated on studying of shear failure mechanisms and specially on modelling the shear failure, see Collins *et al.* (2008), Lee and Kim (2008) or Jensen and Hoang (2009). The shear failure mechanism is a very complex phenomenon. Some experimental studies reveal that unlike flexural failures, reinforced concrete shear failures may be relatively brittle and for members without shear reinforcement it can occur without warning, Collins *et al.* (2008). Nevertheless, the most recent formulated design models for shear in reinforced concrete beams assume plastic effects in steel and concrete.

The shear design method in Eurocode 2 is based on the inclined strut method, which is a method using the theory of plasticity and developed in Denmark in the late 70s, see Nielsen (1998), where plastic design of reinforced concrete is treated in general and for a lot of problems.

The modified compression field theory, originally developed by Vecchio and Collins (1986) is also based on the theory of plasticity and it fulfils equilibrium and compatibility conditions. To make it suitable for design without computers a simplified theory was proposed, Bentz *et al.* (2006). It is used for shear design in AASTHO's Bridge Code (2007).

A further development of the inclined strut method takes place in Denmark. It is the crack sliding theory, which distinguishes between the sliding (shear) failure in cracked and non-cracked concrete [Hoang and Nielsen (1998) or Jensen and Hoang (2009)]. For the latest news in application of the theory of plasticity to concrete, see Proceedings of the Morley Symposium on Concrete Plasticity and its Applications (2007).

In the theory of plasticity we find values for the carrying capacity, which are lower than or equal to yield load by creating stress fields, which fulfil the equilibrium conditions and are safe according to the failure criterions for the materials, see Kwiecinski (1986). Such solutions are named **lower bound solutions**.

We also could find solutions, which are greater than or equal to the yield load by creating failure mechanisms and using the work equation on the mechanisms. Such solutions are named **upper bound solutions**.

In some cases it is possible to find a lower bound solution, which is equal to an upper bound solution and we have an exact solution – the yield load or the carrying capacity.

Failure criterion (=yield criterion) for concrete is taken as the Couloms failure criterion, but in reinforced concrete the tensile strength of the concrete is taken as zero, which means that in plane stress fields, the failure criterion is equal to the square failure criterion (Fig. 1).

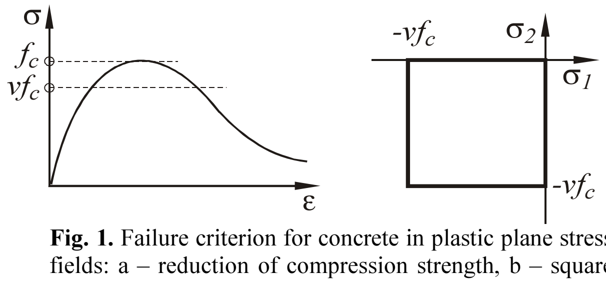

Fig. 1. Failure criterion for concrete in plastic plane stress fields: a – reduction of compression strength, b – square criterion

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Concrete is not an ideal plastic material, which means that a reduced plastic strength νf_c is introduced. The reduction factor ν is called the effectiveness factor, as it is a measure of the effectiveness of the concrete at plastic design (Fig. 1a). Some informations on ν are given in Lapko and Jensen (2005) or Jensen and Lapko (2008).

For the reinforcement it is supposed that only it is able to resist forces in the direction of the bars and the capacity is equal to the yield stress of the reinforcement.

Lines of discontinuity in deformations are a useful method, when creating failure mechanisms for upper bound solutions. For such lines the dissipation (internal work) are developed for materials following Coulomb's failure criterion (Jensen 1975), but here we will only use the lines of discontinuity together with the failure criterion (Fig. 2). In this case, the internal work per unit length and thickness in the concrete is

$$W_l = \frac{1}{2} V \nu f_c (1 - \sin\alpha), \tag{1}$$

where V – the deformation along the line (Fig. 2), where part I of the concrete is moved the distance V relatively to part II.

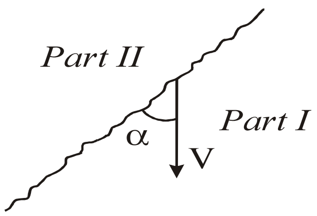

Fig. 2. Deformation along a line of discontinuity

If any reinforcement bars are crossing the line of discontinuity, the work from the reinforcement has to be added to the work from the concrete, see later.

We will have a close look on the classical shear test in beams as shown in Fig. 3. The bending resistance of the beam is so big, that the failure will be a shear failure, which will be in one of the parts of the beam between the supports and the loads.

First we will show, that for this load case we can find the exact plastic load carrying capacity and then we will extend the lower bound method to be a general design method.

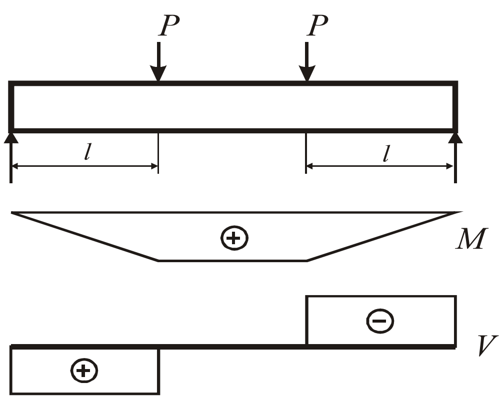

Fig. 3. Beam in the classical shear test

It is clearly shown, that for this load case we can find the exact plastic load carrying capacity and then extend the lower bound method to be a general design method.

2. Solution on the basis of theory of plasticity

2.1. A lower bound solution

We are looking at the shear span from Fig. 3 only. In this area we assume the following stress distribution:

Tension and compression areas are assumed to be stringers, e.g. without any height (Fig. 4). The distance between the stringers z = internal lever arm.

The concrete between the stringers is in a uniaxial compression with the stress σ_c, forming an angle θ with the horizontal stringers. The distance between stirrups is supposed to be so small, that the force in the stirrups can be regarded as a distributed vertical stress σ_{sy}. (Only vertical stirrups are considered).

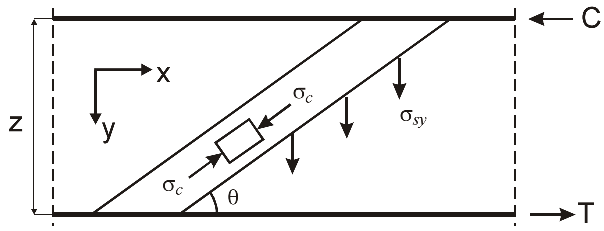

Fig. 4. Stresses and forces in the shear span

In the concrete the principal stresses are 0 and $-\sigma_c$ as stresses are now positive as tensile stresses.

In the shown coordinate system, the concrete stresses then are

$$\sigma_{cx} = -\sigma_c \cos^2\theta, \tag{2}$$

$$\sigma_{cy} = -\sigma_c \sin^2\theta, \tag{3}$$

$$\tau_{cxy} = \sigma_c \sin\theta\cos\theta. \tag{4}$$

The forces in the stirrups are distributed as stresses and they are

$$\sigma_{sx} = \tau_{sxy} = 0, \tag{5}$$

$$\sigma_{sy} = \frac{A_s \sigma_s}{b s}, \tag{6}$$

where: A_s – the area of one stirrup, σ_{sy} – the stress in the stirrups, b – the width of the beam, s – the distance between stirrups.

The total stress distribution between the stringers is found by adding the stresses from the concrete and the stresses from the reinforcement (stirrups)

$$\sigma_x = -\sigma_c \cos^2\theta, \tag{7}$$

$$\sigma_y = -\sigma_c \sin^2\theta + \frac{A_s \sigma_s}{b s}, \tag{8}$$

$$\tau_{xy} = \sigma_c \sin\theta\cos\theta. \tag{9}$$

In the beam we now have a look on a section perpendicular to the *x*-axis, as shown in Fig. 5, where the stringer forces, stresses and the sectional forces are shown.

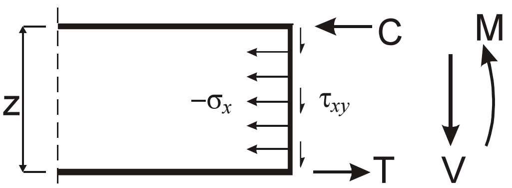

Fig. 5. Stresses and forces in the beam

Vertical equilibrium means

$$\tau_{xy} = \frac{V}{b\,z}. \quad (10)$$

Equilibrium in moments on the tension stringer gives:

$$M = C\,z - \sigma_x \frac{1}{2} z^2 b. \quad (11)$$

From (9) and (7) we get

$$\tau_{xy} \cot\theta = -\sigma_x. \quad (12)$$

Introducing (12) and (10) into (11), we get

$$C = \frac{M}{z} - \frac{1}{2} V \cot\theta. \quad (13)$$

In a similar way we find

$$T = \frac{M}{z} + \frac{1}{2} V \cot\theta. \quad (14)$$

We note that in the shear span the tension force from the moment is increased by the shear and the compression force from the moment is decreased.

For the stress distribution between the stringers (7)–(10) we have the condition $\sigma_y = 0$, as we have no external stresses on top or bottom of the beam. Introducing $\sigma_y = 0$ into (8), we find

$$\tau = \tau_{xy} = \frac{A_s \sigma_s}{b\,s} \cot\theta. \quad (15)$$

From (9) we get

$$\sigma_c = \tau(\tan\theta + \cot\theta). \quad (16)$$

The degree of shear reinforcement is introduced as

$$\psi = \frac{A_s f_y}{b\,s\,f_c}. \quad (17)$$

A lower bound is found by putting the reinforcement stress equal to the yield strength of the reinforcement e.g. $\sigma_s = f_y$, and the concrete stress equal to the plastic strength, e.g. $\sigma_c = \nu f_c$. Introducing these into (15) and (16) and solving with respect to τ and θ, we find

$$\frac{\tau}{f_c} = \sqrt{\psi(\nu - \psi)}, \quad (18)$$

$$\tan\theta = \sqrt{\frac{\psi}{\nu - \psi}}. \quad (19)$$

The solution is valid as long as $\psi \leq \frac{1}{2}\nu$. For higher degree of shear reinforcement, the carrying capacity is

$$\frac{\tau}{f_c} = \frac{1}{2}\nu, \quad \text{when} \quad \psi \geq \frac{1}{2}\nu. \quad (20)$$

To explain the above given limit of ψ we can rearrange (18) into the form

$$\left(\frac{\tau}{f_c}\right)^2 + \left(\psi - \frac{\nu}{2}\right)^2 = \left(\frac{\nu}{2}\right)^2, \quad (21)$$

which is the equation for a circle in a $\frac{\tau}{f_c}$, ψ – coordinate system with its centre in $\left(\frac{\tau}{f_c}, \psi\right) = \left(0, \frac{\nu}{2}\right)$ and radius equal to $\frac{\nu}{2}$. The circle described by the equation (21) is shown in Fig. 6. When the degree of shear reinforcement is greater than ψ = ½ *ν*, it means that the shear sector of the beam is overreinforced.

This above solutions have been validated by the experimental works prepared in Denmark and presented by Nielsen *et al.* (1978). The 198 shear test results conducted on RC simple supported T – beams showed a good applicability of the plastic theory in shear design.

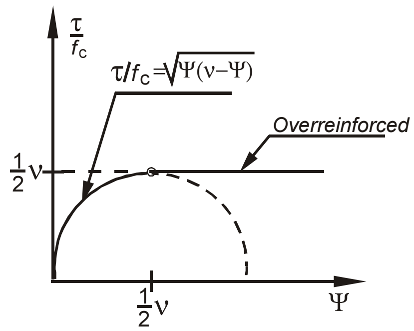

Fig. 6. The plastic solution for the case of beam shown in Fig. 3

2.2. An upper bound solution

Again we consider a beam in shear with tension and compression as stringers. The beam has the shear reinforcement degree ψ and the geometry is shown in Fig. 7 together with yield lines at failure. The failure mechanism is a vertical moving *V* of part I to the two parts II. The moving takes place in the shown straight yield lines, which are forming the angle *β* with the horizontal axis.

As moving is vertical, the forces in the stringers do not contribute to the internal work in the work equation. This work equation for one of the yield lines gives:

$$P\,V = A_s f_y \frac{h \cot\beta}{s} V + \frac{V\,\nu\,f_c\left(1 - \sin(90 - \beta)\right)h}{2\sin\beta}. \quad (22)$$

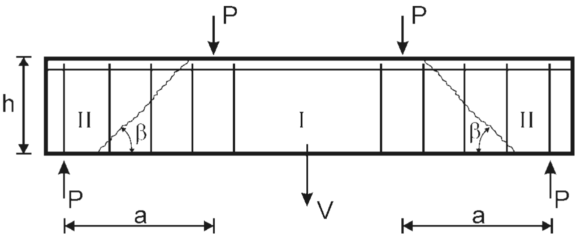

Fig. 7. Beam with shear failure mechanism

The first term on the right-hand side is the dissipation from the stirrups crossing the yield line, and the second term is the dissipation from the concrete using (1).

The upper bound solution can be written:

$$\frac{\tau}{f_c} = \psi \cot\beta + \frac{1}{2}\nu\left(1-\cos\beta\right)\frac{1}{\sin\beta}. \quad (23)$$

The lowest upper bound is found by minimizing (23) with respect to β, and we get

$$\frac{\tau}{f_c} = \sqrt{\psi(\nu-\psi)}, \quad (24)$$

$$\tan\beta = \frac{2\sqrt{\psi(\nu-\psi)}}{\nu-2\psi}. \quad (25)$$

It is noted that the expression (24) is equal to (18), which means, that the formula gives the exact plastic carrying capacity. Furthermore, it is only valid when $\psi \le \frac{1}{2}\nu$, see explanation in Fig. 6. For a bigger amount of shear reinforcement (20) is also valid for the upper bound solution.

Some limitations due to the width of the shear span etc are not dealt with here, but more it can be found in (Nielsen (1998), where also some discussion on other load cases and comparisons with tests can be found.

However important is, that the solution is exact, which means that comparisons with tests make it possible to investigate the effectiveness factor. This factor is subject to national determination, but in Eurocode 2 a safe value is recommended:

$$\nu = 0.6\left(1-\frac{f_{ck}}{250}\right), \ f_c \text{ in MPa}. \quad (26)$$

Because we have an exact solution, it is possible to study the parameters influencing the effectiveness factor ν. Such studies show that the dominating parameter is the concrete strength – simply: concrete shows a decreasing ductility with an increasing strength. Already in the work of Nielsen *et al.* (1978) such analysis was presented showing

$$\nu = 0.8 - \frac{f_c}{200}, \ f_c \text{ in MPa}, \quad (27)$$

but a safe value was recommended to be

$$\nu = 0.7 - \frac{f_c}{200}, \ f_c \text{ in MPa}. \quad (28)$$

This recommended value has been used in Danish codes, also in the Danish National Annex to Eurocode 2 (2007).

3. A practical design method

3.1. The distance between stirrups

Only in special cases an exact value of the capacity can be found. For a general design method the lower bound solution is used as the basis.

We consider a case with an end of a beam with a distributed load on the top of it. The compression and the tension again are stringer forces and the concrete between the stringers has a uniaxial stress σ_c, forming the angle θ with the horizontal axis. We make a cut parallel to the direction of the concrete stress (Fig. 8). No concrete stresses are crossing the cut, only the forces in the vertical stirrups with the total force *N* cross the line.

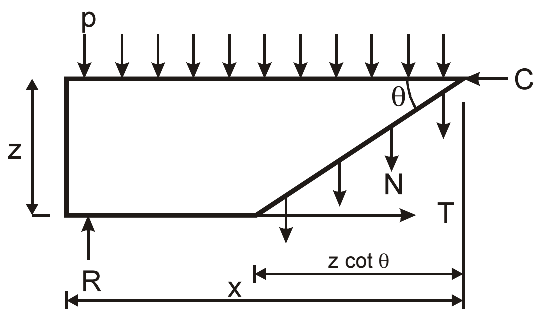

Fig. 8. A beam end with inclined cut along the concrete stresses

From (16) the concrete stress is known and it has to be less than the plastic strength of the concrete, e.g.:

$$\sigma_c = \tau\left(\tan\theta + \cot\theta\right) \le \nu f_c, \quad (29)$$

where

$$\tau = \frac{V}{b\,z}. \quad (30)$$

Theoretically θ can be chosen arbitrarily as long as (29) is satisfied. However, a proper behaviour of the service load requires limitations. The experience of the tests has lead to the following recommendations:

$$1 \le \cot\theta \le 2.5. \quad (31)$$

The limitations were introduced for the first time and the inclined strut method was presented in a code-like edition, see Nielsen and Bach (1980).

Vertical equilibrium of the forces shown in Fig. 8, presents:

$$N = R - p\,x = V_x, \quad (32)$$

where V_x – the shear force at the distance *x* from the reaction.

The area of a stirrup is noted A_{st} and the number crossing the inclined line is noted *n*. We can now write:

$$N = n\,A_{st} f_y = \tau_x b\,z. \quad (33)$$

The distance between stirrups is noted a_t and it is

$$a_t = \frac{z \cot\theta}{n}. \tag{34}$$

Now the distance between stirrups with the cross section area A_{st} can be found in (33) and (34):

$$a_t = \frac{A_{st} f_y}{\tau_x b} \cot\theta. \tag{35}$$

It is noted that within the distance $z\cot\theta$ the shear reinforcement is found from the smallest shear stress.

3.2. Discussion

It is important to note, that the theory of plasticity is dealing with the ultimate load. The calculations are restricted to a situation where failure may develop. Thus the serviceability of load situation is not covered by the calculations.

Looking at the two-point loaded beam with the exact solution, the situation for a growing load may be like this:

After cracking in bending the increasing load will create some cracks in the shear span. They will be with inclination of about 45°, because between compression and tension from bending we have only shear stresses in the uncracked concrete, and after cracking we will have an inclined concrete strut with the inclination of 45°.

An increasing load will increase the inclination of the concrete strut (cotθ increases) and some of the shear reinforcement will yieldi – but the concrete stresses in the strut will still be less than the plastic concrete strength.

Further increase of the load will increase the inclination of the strut, it means that cotθ will increase more, the concrete stresses in the strut will increase and more of the shear reinforcement will yieldi. At the end we will have a failure, when the inclination is increased to a value, where the concrete stresses are equal to the plastic concrete strength νf_c. At this state we will have a failure, with compression failure in the concrete strut and yielding in the shear reinforcement.

It has to be noted that failure means yielding in shear reinforcement and compression failure in the inclined concrete strut at the same time.

For small degrees of shear reinforcement, cotθ will be rather high, see formula (19). When this method for calculating shear was developed in Denmark in the mid 70's, it was noted that very high values of cotθ was followed by big shear crack widths. The lower bound method was introduced in the Danish Concrete Code (1984), but to limit the crack width problem the value of cotθ was limited to 2.5. The method and the limitation on cotθ is introduced in Eurocode 2 (2004).

Thus, the limit $\cot\theta = 2.5$ is a limit purely based on the serviceability state and is not a limit of the carrying capacity. Shear failure only appears when both concrete strut and shear reinforcement fail. With small degrees of shear reinforcement, the limit in design $\cot\theta = 2.5$ does not mean that the carrying capacity is reached, but the crack widths in the serviceability state will be acceptable normally.

It is also to be noted, that the failure needs a good deal of yielding in the shear reinforcement, especially at low degrees of shear reinforcement. We need "translation capacity" of the structure, similar to "rotation capacity" of structures, when plastic design of beams and frames are used. Eurocode 2 limits the reinforcement to class B and class C, when beams and frames are designed according to the theory of plasticity. For plastic design of shear such limitations on the use of reinforcement is not included in Eurocode 2, but a previous research has shown more about necessary translation capacity; it is recommended to restrict the use of shear reinforcement to class B and class C, as it is done in the Danish National Annex to Eurocode 2 (2007). Usage reinforcement in class A is allowed in Denmark, but only with $\cot\theta = 1$ in shear.

To utilize higher values of cotθ for more accurate methods of calculation of shear crack widths are needed and to be more specific in translation capacity we also need more research work.

3.3. Shear crack width design

Prior to inclined cracking, strains in the vertical stirrups are equal to strains in concrete, therefore the stresses in stirrup legs are relatively small. Thus the stirrups cannot prevent the shear zone against an inclined crack appearance. After inclined cracks occur, stirrups come into play in this region. In a flexural member with stirrups these cracks are noted as flexure – shear cracks. The forces in a beam with stirrups and in flexure – shear cracks are presented in Fig. 9 (after MacGregor and Wight 2005).

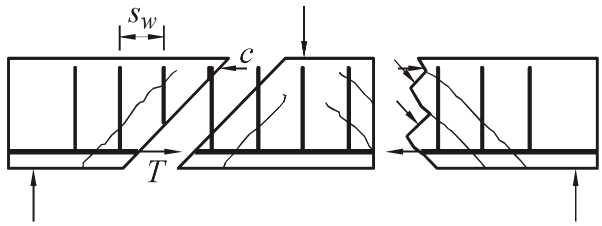

Fig. 9. Inclined shear in a beam

The inclined crack width cannot be predicted by calculating the principal stresses in an uncracked beam. Their slope, spacing and width depend on many factors like flexural and shear reinforcing steel areas, shape and dimension of cross-section, shear stresses and mechanics properties of concrete and steel. For this reason the control of inclined cracking width can be performed using empirical equations, based on experimental works only.

The overview of methods useful for calculating the shear cracks width can be found in Godycki-Ćwirko (1992). On the basis of their experimental studies Placas and Regan (1971) proposed the following formula for evaluation of inclined crack width for a beam with vertical stirrups:

$$w_{\max} = k \frac{s_w}{a_{st}\sqrt[3]{f_{ck}}} \cdot \frac{V_{Sd} - V_{R,cr}}{db_w}, \qquad (36)$$

where k – an empirical coefficient; s_w – spacing of stirrups; V_{Sd} – shear force; $V_{R,cr}$ – shear capacity at the moment of inclined crack appearance, $a_{st} = A_{sw}/b_w$.

Based on general crack analysis, Borishansky (Боришанский 1964) assumed, that the mean inclined crack width w_m depends on strains in stirrups ε_{sm} and crack spacing λ, according to formula

$$w_m = \varepsilon_{sm}\lambda\,. \qquad (37)$$

The mean strains in vertical stirrups were estimated as equal to

$$\varepsilon_{sm} = \frac{f_{ywk}}{E_s}\left(\frac{V_{Sd}}{V_{Rd}}\right)^2, \qquad (38)$$

where: V_{Sd} – shear force; V_{Rd} – shear capacity in function of characteristic strength of steel and concrete.

The inclined crack spacing in a beam with vertical stirrups with diameter ϕ_s, after Borishansky's proposition is calculated as follows

$$\lambda = \frac{1}{\alpha\left(\dfrac{\rho_{ws}\tau_{ms}}{\phi_s f_{ctm}}\right)}, \qquad (39)$$

where:

ρ_{ws} – shear reinforcement ratio $\rho_{ws} = \dfrac{A_{sw}}{b_w s_w}$;

τ_{ms} – mean bond stress at the contact of stirrups and concrete;

f_{ctm} – mean tensile strength of concrete.

The ratio τ_{ms}/f_{ctm} depends on type of steel bar surface (=1.0 for plain bars and 0.7 for ribbed bars),

After Borishansky, the coefficient α = 4; however, in recent Polish Concrete Standard (2002), where the method is implemented, this coefficient has been stated as equal to 3.

Shear capacity V_{Rd} has been expressed as

$$V_{Rd} = \sqrt{4\beta_s f_{ck} b_w d^2 \frac{A_{sw} f_{ywk}}{s_w}}, \qquad (40)$$

where β_s – an empirical coefficient.

Final formula, applied in Polish Concrete Standard (2002) for a member with vertical stirrups is

$$w_m = \frac{4\tau_{ms}^2}{\rho_{sw} E_s f_{ck}}\lambda\,. \qquad (41)$$

This formula may be modified taking into account, instead of shear capacity given in formula (40), the expression derived on the basis of plastic truss model for shear

$$V_{Rd} = \frac{A_{sw}}{s_w} z f_{ywk} \cot\theta, \qquad (42)$$

and on this basis the inclined crack width is equal to

$$w_m = \frac{\tau_{ms}^2 \lambda}{E_s f_{ywk} \rho_{sw} \cot^2\theta}. \qquad (43)$$

Recent studies (Khalfallah 2008) show that for the cracks width in RC members an important role plays the interaction between shear reinforcement and concrete, appearing from bond stress distribution.

3.4. Some aspects in the practical design

For practical design, the beam is distributed in sections with the length $z\cot\theta$ (Fig. 10a) and in each of the section the distance between the stirrups is determined by the smallest shear stress according to (35). For execution on the site this solution may produce too many different distances between stirrups. In such cases a longer length for the same distance between stirrups is chosen and the distance between stirrups is determined by the shear stress at the distance $z\cot\theta$ from the highest value (Fig. 10b). These rules are only valid for distributed load on the top of a beam.

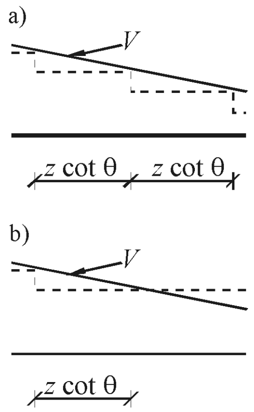

Fig. 10. Beams split into sections with the same distance between stirrups

Design of shear reinforcement in beams with vertical stirrups and without single forces, step by step could be recommended as follows:

1. Design a beam for bending.
2. Find internal lever arm z (distance between compressive C and tensile T forces).
3. Draw the shear stress envelope from the load and shear force, $\tau_{Sd} = V_{Sd}/(zb_w)$.
4. Find the value of $V_{Rd,c}$ according to formulae given in Eurocode 2, if $V_{Sd} > V_{Rd,c}$ calculation of stirrups is obligatory.
5. Choose the concrete compressive strut inclination: $1 \le \cot\theta \le 2.5$.
6. Check the concrete stress;

$$\sigma_c = \tau_{Sd,\max}(\cot\theta + \tan\theta) \le \nu f_{cd}\,.$$

7. Find the distance a_w, where the smallest shear stress can be used for design, $a_w = z\cot\theta$.
8. Choose the stirrup dimension and find the maximum distance $s_{w,\max}$ between stirrups belonging to minimum shear reinforcement (Lee

and Kim studies, 2008). According to Eurocode 2 (2004), provisions it is equal to:

$$s_{w,\min} \le \begin{cases} 0.75d \\ 12.5\dfrac{A_{sw}}{b_w}\dfrac{f_{yk}}{\sqrt{f_{ck}}} \end{cases}.$$

9. Choose in respecting step 8 $s_{sw,\min}$ and find the shear capacity τ_{Rd} belonging to $s_{sw,\min}$,

$$\tau_{Rd,\min} = \frac{A_{sw} f_{ywd}}{s_w b_w}\cot\theta .$$

10. Find where $\tau_{Sd} = \tau_{Rd,\min}$ and the distance, where minimum reinforcement can be used; remember to add the distance $z\cot\theta$ to the point, where $\tau_{Sd} = \tau_{Rd,\min}$.
11. Find the value of τ_{Sd} at the distance 1 from the support and find the belonging s_w.
12. Find the value of τ_{Sd} at the distance 2 from the support and find the belonging s_w

(and so one, until gab to the minimum reinforcement is closed) (Fig. 11a and 11b).

With single forces the distance $a_w = z\cot\theta$ does not have to cross the single force (Fig. 11).

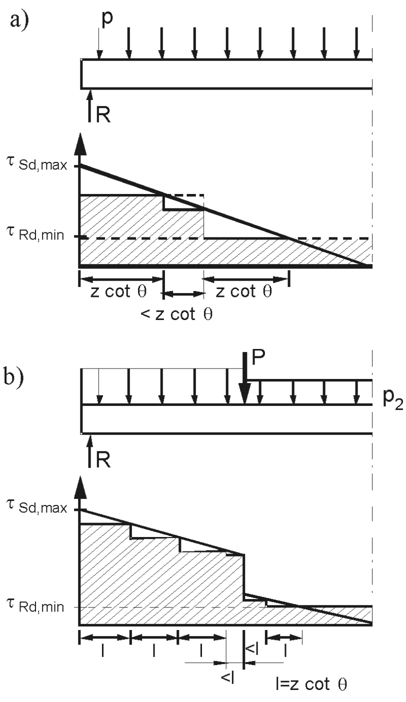

Fig. 11. Cases of design for shear in the beam with single forces

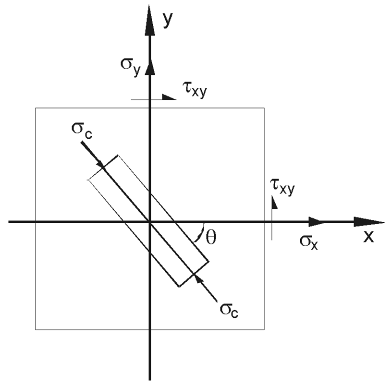

Fig. 12. Uniaxial compression in concrete in a panel structure

Using Mohr's circle, the uniaxial stress field is transferred to stresses in the *x, y*- coordinate system

$$\sigma_{cx} = -\sigma_c \cos^2\theta , \tag{44}$$

$$\sigma_{cy} = -\sigma_c \sin^2\theta , \tag{45}$$

$$\tau_{cxy} = \sigma_c \sin\theta \cos\theta . \tag{46}$$

In the shear-panel we also have reinforcement parallel to the axes. The parallel to x-axis the reinforcement has the cross section area A_{sx} at a length z_y and parallel to the y-axis is A_{sy} at a length z_x. The thickness of the panel is b. The stress in the reinforcement is σ_s. The forces in the reinforcement are equivalent to distributed stresses in the panel, and in the x direction we get

$$\sigma_{sx} = \frac{F_x}{bz_y} = \frac{A_{sx}\sigma_s}{bz_y} . \tag{47}$$

And it is similar in the y direction

$$\sigma_{sy} = \frac{A_{sy}\sigma_s}{bz_x} . \tag{48}$$

Then the sum of stresses in concrete and steel bars is:

$$\sigma_x = -\sigma_c \cos^2\theta + \frac{A_{sx}\sigma_s}{bz_y} , \tag{49}$$

$$\sigma_y = -\sigma_c \sin^2\theta + \frac{A_{sy}\sigma_s}{bz_x} , \tag{50}$$

$$\tau_{xy} = \sigma_c \sin\theta \cos\theta . \tag{51}$$

From (50) we get the concrete stresses which have to be less or equal to the plastic strength νf_c of the concrete or

$$\sigma_c = \tau\left(\cot\theta + \frac{1}{\cot\theta}\right) \le \nu f_c . \tag{52}$$

The capacity of the reinforcement is the yielding strength f_y. In (49) and (50) we put $\sigma_s = f_y$, for σ_c we introduce (52). Doing this, we get the necessary reinforcement in each direction:

$$\frac{A_{sx} f_y}{bz_y} = \sigma_{sx} = \sigma_x + \tau_{xy}\cot\theta , \tag{53}$$

4. General shear design

We are looking at a reinforced concrete panel with a homogeneous stress situation $(\sigma_x, \sigma_y, \tau_{xy})$ from an external load. This stress situation will produce stresses in the concrete and the reinforcement, which is placed in the x and y directions. In the concrete, we only accept uniaxial compression stresses. In Fig. 12 such a stress field is shown – a compression stress σ_c in the concrete, forming the angle θ with the x-axis.

$$\frac{A_{sy} f_y}{b z_x} = \sigma_{sy} = \sigma_y + \tau_{xy} \frac{1}{\cot\theta}. \qquad (54)$$

It is noted, that these formulas are given in Annex F to Eurocode 2 (2004). It is also noted that the angle θ may be chosen arbitrarily as long as (52) is fulfilled. It makes it possible to optimise the amount of reinforcement.

A paper on the complete set of formulas were presented in Warsaw by Nielsen (1963) and the work including works on the optimisation was performed by Nielsen in his thesis (1969). Also, the formulas for optimum reinforcement are given in annex F to Eurocode 2 (2004).

And finally it is noted that in the shear span of a beam (Figs 4, 5) we have only vertical reinforcement which means $A_{sx} = 0$ in (49) and (53), thus formulae (49), (50) and (51) are equal to (7), (8) and (9).

5. Summary and conclusions

It has been shown that the inclined strut method for calculating shear in beams is a method based on the lower bound plastic solution, which in some cases are an exact solution. It is also shown that the equations may be found using the general shear design equations as presented in Eurocode 2 (2004).

The angle of the compression concrete strut may be chosen arbitrarily, as long as the concrete stresses are less than or equal to the plastic concrete strength.

It has to be noted, that the failure of reinforced concrete beams needs a good deal of yielding in the shear reinforcement and inclined crack widths, especially at low degrees of shear reinforcement. We need to recognize "translation capacity" of the structure, similar to "rotation capacity" of structures, when plastic design of beams and frames are used. Eurocode 2 limits the reinforcement due to ductility to class B and C steel, when beams and frames are designed using the theory of plasticity. In the plastic design of shear, such limitations in the use of reinforcement is not included in Eurocode 2, but in Denmark the use of reinforcement in class A is limited to $\cot\theta = 1.0$. To understand the upper limit of $\cot\theta$ more accurate methods of calculation of shear crack widths are needed and to realise, if the ductility of class A steel is sufficient for the plastic design of shear, we also need research work on "translation capacity".

The proposed shear crack width design in beams could be a method to find a more rational limitation in inclination of the compression strut.

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GELŽBETONIO SIJŲ SKERSINĖS ARMATŪROS SKAIČIAVIMAS, TAIKANT PLASTIŠKUMO TEORIJOS PRINCIPUS

B. Ch. Jensen, A. Lapko

S a n t r a u k a

Šiuolaikinis gelžbetonio elementų skersinės armatūros skaičiavimas pagrįstas plastiškumo teorijos principais. Straipsnyje pateikti sijų skersinės armatūros skaičiavimo ypatumai, taikant įstrižojo statramsčio modelį, kuris taikomas ir *Eurokode 2*. Išnagrinėtas gniuždomojo strypo pavertimas santvaros modelyje, atsižvelgiant į skersinės armatūros ir efektyvaus gniuždomojo betono stiprio santykį. Aptartas armatūrinio plieno strypų stamantrumas, akcentuota sijų laikomosios galios šlyčiai eksperimentinių tyrimų būtinybė. Pateiktas gelžbetoninių sijų šlyties skaičiavimas, taikant plastiškumo teorijos principus. Aptartas *Eurokode 2* šlyties skaičiavimams taikomų priklausomybių teorinis pagrindas.

Reikšminiai žodžiai: gelžbetoninės sijos, projektavimas šlyčiai, betono plastiškumas, ilginio metodas, įstrižojo plyšio plotis.

Bjarne Christian JENSEN is Professor Em. at the Engineering Faculty, the University of Southern Denmark. Member of CEN/TC 250, Structural Eurocodes and has been a member of SC 2, Concrete Structures. His research is within the area of concrete plasticity and he has taken part in the development in Denmark of plastic design methods for shear in reinforced concrete. Consultant to the Danish Enterprise and Construction Authority; the Danish member of Eurocodes National Correspondents in the EU-Commission and member of the Danish Academy of Technical Science.

Andrzej LAPKO is Professor, many years working at the Bialystok Technical University as the head of the Building Structures Department. Member of Civil and Water Engineering Committee of Polish Academy of Science, the Polish delegate of *fib* and individual member of IABSE. His research interests are in reinforced concrete and composite structures, particularly the structures of silos and water tanks, also thermal loads and effects on concrete structures.