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INCREMENTAL METHOD FOR UNLOADING PHENOMENON FIXATION AT SHAKEDOWN

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Abstract. Incremental method for shakedown analysis of the elastic perfectly plastic structures is based on the extremum energy principles and non-linear mathematical programming approach. Residual force increment calculation problem is developed applying minimum complementary deformation energy principle. The Rozen project gradient and equilibrium finite element methods were applied for solution. The Rozen optimality criterion (Kuhn-Tucker conditions) ensures compatibility of residual strains and allows plastic strain and residual displacement increment calculation without dual problem solution. The possibility to fix the structure cross-section unloading phenomenon during shakedown process was developed. The proposed technique is illustrated by annular bending plate residual force and deflection calculation examples, when the von Mises criterion is taken into account.

Keywords: shakedown, energy principles, mathematical programming, incremental analysis, unloading phenomenon.

1. Introduction

The elastic perfectly plastic structure is considered, the configuration, material, sandwich cross-section dimension of the structure and external load are prescribed. The structure adapted to the cyclic loading satisfies the constraints on strength and it is not likely to undergo cyclic plastic failure [1]. Nevertheless, in the shakedown structure some strains and deflections can appear which do not correspond to the maintenance conditions [2–6]. Hence in the mathematical models of optimization problems for shakedown structures both the strength and stiffness requirements must be taken into account [7–14] (Fig 1). When displacements are not restrained by stiffness conditions, the optimizing structure reaches the limit state related to cyclic-plastic failure.

In structural engineering the stiffness constraints are realized via the displacements (most deflections are constrained). Therefore for a structure under plastic behaviour prior to a cyclic-plastic failure it is necessary to know not only actual stresses but also strains and displacements (structural analysis problem) [15]. A large number of authors directly base their ideas on classical Melan and Koiter shakedown theorems [16] in the structural analysis problems. In this case mathematical programming is applied only as a tool for simple structures in solving the shakedown extremum problem. In the structural shakedown theory and practice every shakedown system under repeated load calculation technique and algorithms creation remains relevant. The quality of those algorithms results in a successful solution of the optimization problem.

In this paper the analysis of problems of dissipative structures (ie structures under plastic deformation) are formulated on the basis of the extremum energy principles characterising the actual structure stress-strain conditions [8, 9]. Using non-linear programming theory, mathematical models of dual stress and strain analysis extremum problems of structure at shakedown are obtained (Fig 1, direct applications of mathematical programming duality theory for structure analysis at shakedown). Dual mathematical programming problems simu-

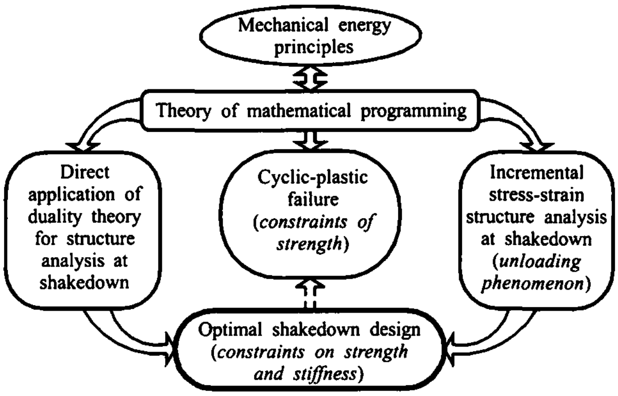

Fig 1. Connections between analysis and optimization problems

late an actual behaviour of structure at shake-down only when there is not any unloading phenomenon in cross-sections [17]. Applying the Rozen criterion [18] for incremental analysis problem solution, a new technique is created to determine unloading phenomenon in cross-sections during adaptation to quasi-static load process (Fig 1, incremental stress-strain structure analysis at shake-down).

2. Discrete definition of elastic-plastic structures

Equilibrium finite element method [19–20] is applied for structure discretisation. Using this method more exact equilibrium equations are obtained to compare with other finite element methods. In company with it statically possible elastic $\mathbf{S}_e$ (subscript e) and residual $\mathbf{S}_r$ forces (total forces are denoted by $\mathbf{S} = \mathbf{S}_e + \mathbf{S}_r$, displacements – $\mathbf{u} = \mathbf{u}_e + \mathbf{u}_r$) are defined more exactly. Structure discrete model degree of freedom is m, vectors of global displacements $\mathbf{u}$ and load $\mathbf{F}$ are $\mathbf{u} = (u_1, u_2, \dots, u_m)^T$ and $\mathbf{F} = (F_1, F_2, \dots, F_m)^T$ respectively. Force vector of element k ($k = 1, 2, \dots, \eta$; $k \in K$) with η_k nodal points ($l = 1, 2, \dots, \eta_k$, $l \in L$) is $\mathbf{S}_k = (\mathbf{S}_{k1}, \mathbf{S}_{k2}, \dots, \mathbf{S}_{kl}, \dots, \mathbf{S}_{\eta_k})^T$. The total component number of vector $\mathbf{S}_k$ is n_k. Forces $\mathbf{S}_k(\mathbf{x})$ at any point $\mathbf{x}$ of finite element are expressed via forces $\mathbf{S}_k$ of element nodal points using matrix of approximation $\mathbf{S}_k(\mathbf{x}) = [N_k(\mathbf{x})]\,\mathbf{S}_k$. The equilibrium of discrete model is ensured only for structure elements and their main nodes [20]. Finally, taking into account boundary conditions, the structure equilibrium equations, system reads:

$$[A]\mathbf{S} = \mathbf{F} \text{ or } \sum_k [A_k]\mathbf{S}_k = \mathbf{F}, \tag{1}$$

where $[A]$ is $m \times n$ the equilibrium matrix. The statically possible residual forces $\mathbf{S}_r$ are self-balanced:

$$[A]\mathbf{S}_r = 0. \tag{2}$$

Geometric equations for structure discrete model read:

$$[A]^T\mathbf{u} - [D]\mathbf{S} = 0. \tag{3}$$

Here $[D] = diag\,[D_k]$ is the quasidiagonal structure flexibility matrix ($k \in K$). Element flexibility matrix $[D_k]$ in the local coordinates is obtained applying the formula

$$[D_k] = \int_{A_k} [N_k(\mathbf{x})]^T [\mathcal{D}][N_k(\mathbf{x})]dA, \quad k \in K.$$

The physical meaning of displacement vector $\mathbf{u}$ components is determined by equilibrium equations (1) formation order and dual relationship between equilibrium equations (1) and geometric ones (3). Applying the known finite elements procedures, elastic displacements $\mathbf{u}_e$ and forces $\mathbf{S}_e$ are obtained. Kinematically possible residual displacements $\mathbf{u}_r$ satisfy geometric equations (3):

$$[A]^T\mathbf{u}_r = \mathbf{\Theta}_r, \quad \mathbf{\Theta}_r = [D]\mathbf{S}_r + \mathbf{\Theta}_p, \tag{4}$$

where $\mathbf{\Theta}_p$ is the vector of plastic strains. Residual strains $\mathbf{\Theta}_r$ and displacements $\mathbf{u}_r$ can be non-unique: they depend on the particular loading history $\mathbf{F}(t)$.

It is difficult to take into account loading history, when loading $\mathbf{F}(t)$ is described via time t, independent load variation bounds $\mathbf{F}_{sup}$ and $\mathbf{F}_{inf}$ ($\mathbf{F}_{inf} \le \mathbf{F}(t) \le F_{sup}$). For instance, two loads $F_1(t)$, $F_2(t)$ variation field is shown as a dark quadrangle in Fig 2. Here the number of external force $\mathbf{F}_{inf} = (F_{1,inf}, F_{2,inf})^T$, $\mathbf{F}_{sup} = (F_{1,sup}, F_{2,sup})^T$ combinations $j = 1, 2, \dots, p$, $p = 4$ ($p = 2^m$, $m = 2$). During residual stresses and displacement analysis the problem of structure at shakedown solution vectors $\mathbf{F}_{sup}$ and $\mathbf{F}_{inf}$ must be prescribed, because of that quadrangle shown in Fig 2 is a constant form. This feature is applied because of duality theory direct application to structural analysis at shakedown. However, it is possible to take into account possible loading history, when only load variation bounds are given. It was achieved in this paper in two different ways. First, sequentially load variation field was extended in line with arbitrary increment (there are quadrangles marked by dotted lines in Fig 2). That is not an exact loading history evaluation. In residual stresses and displacements analysis problems particular loading history $\mathbf{F}(t)$ is evaluated more exactly if vector $\mathbf{F}$ components – forces assume increments ΔF, but not their variation field. One of many possible histories $\mathbf{F}(t)$ is shown as a continued line in Fig 2. Such loading history evaluation possibilities when solving actual residual forces $\mathbf{S}_r$ and displacements $\mathbf{u}_r$ calculation questions are considered in mathematical model formation for incremental stress-strain analysis problems at shakedown (Fig 1). Only in this way it is possible to take into account

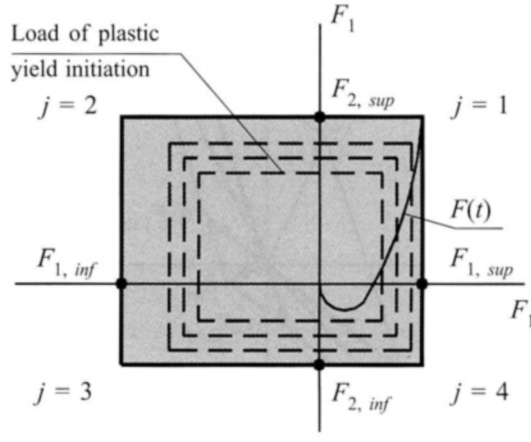

Fig 2. Two independent force variation field

the unloading phenomenon in cross-sections of structure, which is practically necessarily appearing during adaptation.

Actually, in engineering practice an influence matrix

$$[\alpha]=[K][A]^{T}\left([A][K][A]^{T}\right)^{-1},\quad [K]=[D]^{-1}$$

to internal forces $\mathbf{S}_e = [\alpha]\,\mathbf{F}$ evaluation is rarely applied. But find that the internal forces influence matrix $[\alpha]$ expressively illustrates an interaction of the internal forces $\mathbf{S}_e$ and load $\mathbf{F}$ when formulating the mathematical model of structure analysis and load optimization problems. Let us say that the actual load process is described via the time t independent of load variation bounds $\mathbf{F}_{inf}$ and $\mathbf{F}_{sup}$ ($\mathbf{F}_{inf} \le \mathbf{F}(t) \le F_{sup}$). Then the certain distribution $\mathbf{S}_{ej}$ is calculated for each external forces combination j (ie for the vector of loads variation bounds components $\mathbf{F}_{inf}$, $\mathbf{F}_{sup}$), all combinations being coupled to the set $j = 1,2,\dots, p$, $j \in J$. The elastic solution vectors $\mathbf{S}_{ej}$ are linear functions of the load variation bounds $\mathbf{F}_{inf}$, $\mathbf{F}_{sup}$ and define all vertices $j \in J$ of elastic force locus

$$\mathbf{S}_e(t) = [\alpha]\ \mathbf{F}(t). \tag{5}$$

Note that apexes of elastic locus are not located symmetrically regarding the coordinate system origin O (Fig 3). Considering cyclic-plastic failure of the structure it is useful to make out symmetric $\mathbf{S}_e$ apexes. It allows two identification types of failure: progressive plastic failure or an alternating plasticity [9].

Plasticity constant $C = (S_0)^2$ of elastic-plastic structure relates to dimensions and material of ideal form (sandwich) cross-section, ie, to limit force S_0 (Fig 4). Limit force S_{0k} ($k \in K$) is assumed as constant in the whole finite element. Non-linear yield condition

$$\varphi = C - f\left(\mathbf{S}_e(t) + \mathbf{S}_r(t)\right) \ge 0 \tag{6}$$

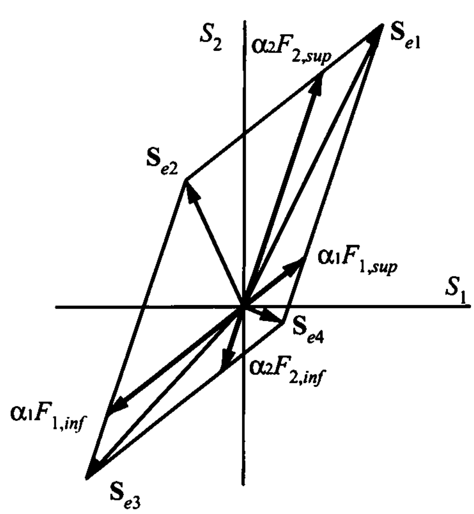

Fig 3. Elastic force locus

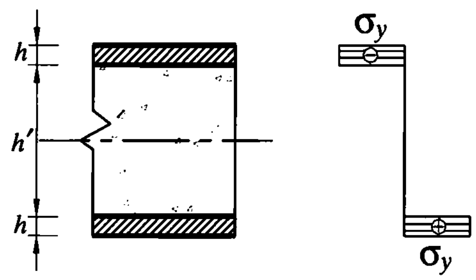

Fig 4. Three-layered bending plate cross-section

is written for that case, when adaptation process is considered in the time t (taking into account all possible loading histories $\mathbf{F}(t)$). Residual forces $\mathbf{S}_r$ of the structure at shakedown must satisfy yield conditions (6) in each cross-section taking into account all apexes j of elastic force locus $\mathbf{S}_e(t) = [\alpha]\ \mathbf{F}(t)$:

$$\varphi_j = C_k - f_j\left(\mathbf{S}_{ej} + \mathbf{S}_r\right) \ge 0,\quad j \in J. \tag{7}$$

These yield conditions are verified in every k finite element nodal point l:

$$\varphi_{kl,j} = C_k - f_{kl,j}\left(\mathbf{S}_{ekl,j} + \mathbf{S}_{rkl}\right) \ge 0,$$
$$k \in K,\ l \in L,\ j \in J. \tag{8}$$

For the entire elastic-plastic structure, using vectors $\mathbf{S}_{ej}$, $j \in J$, the conditions (8) can be rewritten as follows:

$$\varphi_j = \mathbf{C} - \mathbf{f}_j\left(\mathbf{S}_{ej} + \mathbf{S}_r\right) \le \mathbf{C},\quad j \in J. \tag{9}$$

Here $\mathbf{C} = \left(C_1,\ C_2,\dots, C_k,\dots, C_\eta\right)^T$ is a vector of the whole structure plasticity constants. Hubert-Mises yield condition will be applied during consideration of elastic-plastic plates. Statically admissible residual stresses $\mathbf{S}_r$ satisfy equilibrium equations (2) and yield conditions (9).

3. Direct applications of duality theory for structure analysis at shakedown

3.1. Static analysis problem formulation

Residual force vector $\mathbf{S}_r$ for structure at shakedown is obtained by solving static analysis problem formulation. This formulation is made on the basis of the minimum complementary deformation energy principle [7–9]:

of all statically admissible residual forces of structure at shakedown is the minimum complementary energy corresponding one.

The above-mentioned principle leads to the extremum problem as follows:

$$\text{minimise}\ \frac{1}{2}\sum_k \mathbf{S}_{rk}^T[D_k]\ \mathbf{S}_{rk} = a^*, \tag{10}$$

subject to

$$\sum_k [A_k]\mathbf{S}_{rk} = 0, \qquad (11)$$

$$\mathbf{S}_{rk} = \left(\mathbf{S}_{rk1}, \mathbf{S}_{rk2}, ..., \mathbf{S}_{rkl}, ..., \mathbf{S}_{rk\eta_k}\right)^T, \qquad (12)$$

$$\varphi_{kl,j} = C_k - f_{kl,j}\left(\mathbf{S}_{ekl,j} + \mathbf{S}_{rkl}\right) \geq 0, \qquad (13)$$

$$C_k = (S_{0k})^2,\ k \in K,\ l \in L,\ j \in J, \qquad (14)$$

where structure limit force vector $\mathbf{S}_0 = \left(S_{01}, S_{02}, ..., S_{0k}, ..., S_{0\eta}\right)^T$ is known. In yield conditions $\varphi_{kl,j} = C_k - f_{kl,j}\left(\mathbf{S}_{ekl,j} + \mathbf{S}_{rkl}\right) \geq 0$ all apexes $p,\ j \in J$ of elastic force locus (5) $\mathbf{S}_e(t) = [\alpha]\ \mathbf{F}(t)$ are taken into account. The functions $\varphi_{kl,j} \geq 0$ are convex, the matrix $[D_k]$ is positively defined, therefore optimal solution of analysis problem (10)–(14) is global and noted by $\mathbf{S}_r^*$. The adaptation of the structure is caused not by a minimum value of complementary deformation energy, but by the fact that there do not exist statically admissible, ie satisfying equilibrium equations (11), yield conditions (13)–(14) and residual forces $\mathbf{S}_r$ [21, 22].

3.2. Kinematic analysis problem formulation

Residual displacements $\mathbf{u}_r$ of the structure at shakedown are obtained by solving dual problem to the initial one (10)–(14):

maximise

$$\left\{-\frac{1}{2}\sum_k \mathbf{S}_{rk}^T[D_k]\,\mathbf{S}_{rk} - \sum_k\sum_j \lambda_{k,j}^T\left[\nabla\varphi_{k,j}\left(\mathbf{S}_{k,j}\right)\right]\mathbf{S}_{rk} - \right.$$

$$\left. - \sum_k\sum_l\sum_j \lambda_{kl,j}\left[C_k - f_{kl,j}\left(\mathbf{S}_{kl,j}\right)\right]\right\}, \qquad (15)$$

subject to

$$[D_k]\,\mathbf{S}_{rk} + \sum_j\left[\nabla\varphi_{k,j}\left(\mathbf{S}_{k,j}\right)\right]^T\lambda_{k,j} - [A_k]^T\mathbf{u}_r = 0, \qquad (16)$$

$$\mathbf{S}_{k,j} = \left(\mathbf{S}_{ek,j} + \mathbf{S}_{rk}\right),\ \ \lambda_{k,j} \geq 0,$$

$$k \in K,\ l \in L,\ j \in J. \qquad (17)$$

Components of plastic strain vector $\Theta_p = \left(\Theta_{pkl}\right)^T$, included in geometric equations, are determined by formula:

$$\Theta_{pkl} = \sum_j \Theta_{pkl,j} = \sum_j\left[\nabla\varphi_{kl,j}\left(\mathbf{S}_{ekl,j} + \mathbf{S}_{rkl}\right)\right]^T\lambda_{kl,j},$$

$$\lambda_{kl,j}\,\varphi_{kl,j} = 0,\ \lambda_{kl,j} \geq 0;$$

$$k \in K,\ l \in L,\ j \in J. \qquad (18)$$

Here $\left[\nabla\varphi_{kl,j}\left(\mathbf{S}_{ekl,j} + \mathbf{S}_{rkl}\right)\right]^T = \left[\dfrac{\partial f_{kl,j}\left(\mathbf{S}_{ekl,j} + \mathbf{S}_{rkl}\right)}{\partial \mathbf{S}_{rkl}}\right]$ are gradients of yield conditions (13)–(14), $\lambda_{kl,j}$ – plastic multipliers. In the problem (15)–(17) residual forces $\mathbf{S}_r$, displacements $\mathbf{u}_r$ and plasticity multipliers λ_j, $j \in J$ are assumed as unknowns. By changing the sign of the objective function (15) to the opposite, the following extremum energy principle is obtained:

of all kinematically admissible residual displacement distributions, the actual one corresponds the minimum of total potential energy.

The complementary slackness conditions

$$\lambda_{kl,j}\left(C_k - f_{kl,j}\left(\mathbf{S}_{kl,j}\right)\right) = 0,$$

$$\lambda_{kl,j} \geq 0,\ \ k \in K,\ l \in L,\ j \in J \qquad (19)$$

are incorporated in problems (15)–(17). According to the relations (19) at the moment of plastic strains deformation in the structure j-th cross-section the following relations are valid: $\varphi_{kl,j} = 0$, $\lambda_{kl,j}\,\varphi_{kl,j} = 0$ and $\lambda_{kl,j} > 0$. During structure deformation process, the magnitude of plasticity multiplier $\lambda_{kl,j} > 0$ remains unchanged up to the loading end. The complementary slackness conditions do not allow direct evaluation of the unloading phenomenon (one can meet it when for an actual loading process $\varphi_{kl} = C_k - f_{kl}\left(\mathbf{S}_{c,kl}^{\mathrm{v}} + \Delta\mathbf{S}_{rkl}^{\mathrm{v}}\right) > 0$). Optimal solution $\mathbf{S}_r^*$, $\mathbf{u}_r^*$ and λ_j^* ($j \in J$) of the problem (15)–(17) is obtained without considering the loading history. Nevertheless, a particular loading history exists $\mathbf{F}(t)$ ($\mathbf{F}_{inf} \leq \mathbf{F}(t) \leq F_{sup}$) which leads the structure to shakedown with $\mathbf{S}_r^*$, $\mathbf{u}_r^*$ and λ_j^*. It becomes obvious that the analysis problem mathematical model (10)–(14) of the structure at shakedown serves to structure with holonomic behaviour and can be obtained according to Haar-Kármán principle [16].

Residual displaceme $\mathbf{u}_r$ elimination from equations (16) leads to strain compatibility ones:

$$-[B]\Theta_p = [B_r]\,\mathbf{S}_r. \qquad (20)$$

Here the matrices $[B]$, $[B_r]$ read:

$$[B] = \left[-[A'']^T\left([A']^T\right)^{-1},\ [I]\right], \qquad (21)$$

$$[B_r] = -[A'']^T\left([A']^T\right)^{-1}[D'] + [D'']. \qquad (22)$$

The Rozen project gradient method [17] is known as an algorithm for convex mathematical programming problem solution. The vector $\mathbf{S}_r^*$ is the optimal solution of the problem (10)–(14) if it satisfies the Rozen algorithm optimality criterion [23]. The Rozen optimality criterion coincides with the Kuhn-Tucker conditions known in mathematical programming. The Rozen criterion mathematical-mechanical interpretation: there are strain compatibility equations (20) (together with the complementary slackness conditions (19)) [24, 25]. According to the Rozen criterion for problem (10)–(14) optimal solution plastic multipliers λ^* become known at once. Plastic strains Θ_p are determined from formulas (18). In this case it is not necessary to solve dual analysis problem (15)–(17) (Fig 5). That allows creating practical an iterative solution algorithm for incremental shakedown analysis problem.

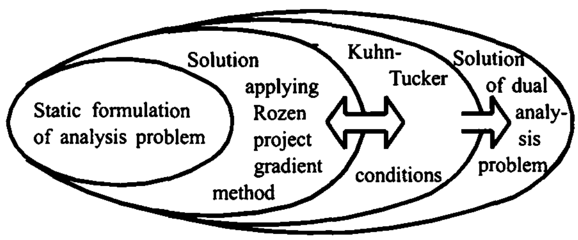

Fig 5. Connections between the Rozen criterion and Kuhn-Tucker conditions

4. Mathematical models of incremental structure analysis at shakedown

4.1. One active force evaluation

Primarily the process of elastic-plastic structure deformation is under consideration, when one load $F(t)$ is acting. For each stage ν ($\nu = 1, 2, \ldots, z, \ \nu \in V$) of calculation process load increment ΔF^{ν} is chosen freely. Sequentially increasing load by this load increment up to its final magnitude $F = \sum_{\nu} \Delta F^{\nu}$, each ν-th stage increments of the residual stress and displacements $\Delta \mathbf{S}^{\nu}$, $\Delta \mathbf{u}^{\nu}$ are determined respectively. At the end of the plastic deformation ν-th stage stress state of structure, discrete model is described by n-vector of total forces $\mathbf{S}^{\nu}$:

$$\mathbf{S}^{\nu} = \mathbf{S}_e^{\nu} + \mathbf{S}_r^{\nu}. \tag{23}$$

here $\mathbf{S}_e^{\nu} = \mathbf{S}_{e\Sigma} + \Delta \mathbf{S}_e^{\nu} = \sum_{\nu} \Delta \mathbf{S}_e^{\nu-1} + \Delta \mathbf{S}_e^{\nu}$ is a vector of pseudo-elastic stresses, and $\mathbf{S}_r^{\nu} = \mathbf{S}_{r\Sigma} + \Delta \mathbf{S}_r^{\nu} = \sum_{\nu} \Delta \mathbf{S}_r^{\nu-1} + \Delta \mathbf{S}_r^{\nu}$ – vector of residual stresses at the end of the ν-th stage. When $\nu = 1$, usually the initial force vectors $\Delta \mathbf{S}_r^{\nu-1} = \Delta \mathbf{S}_r^{0} = \mathbf{0}$, $\Delta \mathbf{S}_e^{\nu-1} = \Delta \mathbf{S}_e^{0} = \mathbf{0}$ (there are no other residual strains, only the F caused ones, in the structure). Components of elastic stress increment vector $\Delta \mathbf{S}_e^{\nu}$ are calculated by the formula

$$\Delta \mathbf{S}_e^{\nu} = \alpha \cdot \Delta F^{\nu}. \tag{24}$$

Here α – column of elastic calculation stress influence matrix $[\alpha]$ related to force F.

At the end of the ν-th stage, total displacements $\mathbf{u}^{\nu}$ are obtained from the relation

$$\mathbf{u}^{\nu} = \mathbf{u}_e^{\nu} + \mathbf{u}_r^{\nu}. \tag{25}$$

Structure elastic displacements are as follows:

$$\mathbf{u}_e^{\nu} = \mathbf{u}_{e\Sigma} + \Delta \mathbf{u}_e^{\nu} = \sum_{\nu} \Delta \mathbf{u}_e^{\nu-1} + \Delta \mathbf{u}_e^{\nu},$$

$$\Delta \mathbf{u}_e^{\nu} = \beta \cdot \Delta F^{\nu}.$$

Here β – column of elastic calculation displacement influence matrix $[\beta]$. At the end of the ν-th stage, residual displacements $\mathbf{u}_r^{\nu}$ are calculated as follows:

$$\mathbf{u}_r^{\nu} = \mathbf{u}_{r\Sigma} + \Delta \mathbf{u}_r^{\nu} = \sum_{\nu} \Delta \mathbf{u}_r^{\nu-1} + \Delta \mathbf{u}_r^{\nu}. \tag{26}$$

The residual displacement increment $\Delta \mathbf{u}_r^{\nu}$ calculation at the end of ν-th stage will be considered later.

Static formulation analysis problem (10)–(14) via residual force increments $\Delta \mathbf{S}^{\nu}$ obtains the following form:

minimise

$$\frac{1}{2} \sum_k \left(\mathbf{S}_{r\Sigma,k} + \Delta \mathbf{S}_{rk}^{\nu}\right)^T [D_k] \left(\mathbf{S}_{r\Sigma,k} + \Delta \mathbf{S}_{rk}^{\nu}\right) = a^{*\nu}, \tag{27}$$

subject to

$$\sum_k [A_k] \Delta \mathbf{S}_{rk}^{\nu} = 0, \tag{28}$$

$$\Delta \mathbf{S}_{rk}^{\nu} = \left(\Delta \mathbf{S}_{rk1}^{\nu}, \Delta \mathbf{S}_{rk2}^{\nu}, \ldots, \Delta \mathbf{S}_{rkl}^{\nu}, \ldots, \Delta \mathbf{S}_{rk\eta_k}^{\nu}\right)^T,$$

$$\varphi_{kl} = C_k - f_{kl}\left(\mathbf{S}_{c,kl}^{\nu} + \Delta \mathbf{S}_{rkl}^{\nu}\right) \geq 0,$$

$$\mathbf{S}_{c,kl}^{\nu} = \mathbf{S}_{e,kl}^{\nu} + \mathbf{S}_{r\Sigma,kl} = \mathbf{S}_{e\Sigma,kl} + \Delta \mathbf{S}_{e,kl}^{\nu} + \mathbf{S}_{r\Sigma,kl},$$

$$\Delta \mathbf{S}_e^{\nu} = \alpha \cdot \Delta F^{\nu}, \ C_k = (s_{0k})^2, \ k \in K, \ l \in L. \tag{29}$$

Problem (27)–(29) unknowns are residual force increment vector $\Delta \mathbf{S}_{rk}^{\nu}$ at the end of ν-th stage (optimal problem solution is noted by $\Delta \mathbf{S}_{rk}^{*\nu}$). Residual forces $\mathbf{S}_{r\Sigma}$, like elastic ones $\mathbf{S}_e^{\nu}$ are known in the problem (27)–(29). At the end of the last stage of loading program elastic forces $\mathbf{S}_e$ are obtained from formula

$$\mathbf{S}_e = \sum_{\nu=1}^{z} \Delta \mathbf{S}_e^{\nu-1} + \Delta \mathbf{S}_e^{z}, \tag{30}$$

and residual ones $\mathbf{S}_r$ – from relation

$$\mathbf{S}_r = \sum_{\nu=1}^{z} \Delta \mathbf{S}_r^{*\nu-1} + \Delta \mathbf{S}_r^{*z}. \quad (31)$$

When $\Delta \mathbf{S}_e^0 = 0$, $\Delta \mathbf{S}_r^0 = 0$, formulas (30), (31) obtain the form

$$\mathbf{S}_e = \sum_{\nu=1}^{z} \Delta \mathbf{S}_e^{\nu}, \ \mathbf{S}_r^* = \sum_{\nu=1}^{z} \Delta \mathbf{S}_r^{*\nu}. \quad (32)$$

Optimal solution $\Delta \mathbf{S}_r^{*\nu}$ is achieved when Rozen optimality criterion (Kuhn-Tucker conditions) [3] is satisfied (calculation schema, shown in Fig 5, serves and for incremental structural analysis at shakedown). Object function $\mathcal{F}$ gradient $\nabla \mathcal{F}$ and gradients of all constraints (equilibrium equations, yield conditions $\varphi \geq \mathbf{0}$) multiplied by Lagrange multipliers λ^{ν} and $\mathbf{u}_r^{\nu}$ are to be calculated in order to write the optimality criterion. Then Kuhn-Tucker conditions are as follows:

$$[D]\Delta \mathbf{S}_r^{*\nu} + [D]\mathbf{S}_{r\Sigma} + [\nabla\varphi]^T \lambda^{*\nu} - [A]^T \mathbf{u}_r^{*\nu} = \mathbf{0}, \quad (33)$$

$$\lambda^{*\nu T}\varphi = \mathbf{0}, \ \lambda^{*\nu} \geq \mathbf{0}, \ \nu \in V. \quad (34)$$

Here $[\nabla\varphi]^T \lambda^{*\nu}$ are total plastic strains at end of the ν-th stage

$$\Theta_p^{*\nu} = [\nabla\varphi]^T \lambda^{*\nu}, \ \lambda^{*\nu T}\varphi = \mathbf{0}, \ \lambda^{*\nu} \geq \mathbf{0}. \quad (35)$$

Physical meaning of Lagrange multipliers: $\mathbf{u}_r^{*\nu}$ – total residual displacements (26) and total plastic multipliers $\lambda^{*\nu} \geq \mathbf{0}$ at the end of ν-th stage.

At the end of the ν-th stage vectors of plastic strain increments $\Delta\Theta_p^{*\nu}$ and residual displacement increments $\Delta\mathbf{u}_r^{*\nu}$ are calculated as follows:

$$\Delta\Theta_p^{*\nu} = \Theta_p^{*\nu} - \Theta_{p\Sigma} = \Theta_p^{*\nu} - \sum_{\nu} \Delta\Theta_p^{\nu-1},$$

$$\Delta\mathbf{u}_r^{*\nu} = \mathbf{u}_r^{*\nu} - \mathbf{u}_{r\Sigma} = \mathbf{u}_r^{*\nu} - \sum_{\nu} \mathbf{u}_r^{\nu-1}. \quad (36)$$

Plastic strains Θ_p^z at the end of loading program read:

$$\Theta_p^z = \sum_{\nu=1}^{z} \Delta\Theta_p^{*\nu}. \quad (37)$$

This formula serves if only before was not any primary plastic strain $\Delta\Theta_p^0 = 0$.

When $\Delta\mathbf{u}_r^0 = 0$, residual displacements at the end of the loading process last stage can be calculated:

$$\mathbf{u}_r^z = [H]\Theta_p^z. \quad (38)$$

Here $[H]$ is the influence matrix of residual displacement:

$$[H] = \left([A][D]^{-1}[A]^T\right)^{-1}[A][D]^{-1}.$$

Problem (27)–(29) optimal solution of each stage $\Delta \mathbf{S}_r^{*\nu}$ can be tested by formula

$$\Delta \mathbf{S}_r^{*\nu} = [G]\Delta\Theta_p^{*\nu},$$

and solution (31) by relation

$$\mathbf{S}_r = [G]\Theta_p. \quad (39)$$

Here

$[G] = [D]^{-1}[A]^T\left([A][D]^{-1}[A]^T\right)^{-1}[A][D]^{-1} - [D]^{-1}$ is the residual force influence matrix.

4.2. Several active force evaluation

The technique proposed in the section 4.1 can be generalised for force increment vector $\Delta\mathbf{F}$ (so can be realized the loading history $\mathbf{F}(t)$ shown as a continuous line in Fig 2). Then formula (24) obtains the form

$$\Delta \mathbf{S}_e^{\nu} = [\alpha]\ \Delta\mathbf{F}^{\nu}. \quad (40)$$

Further adapted structure analysis problem solution is performed according to in the section 4.1 explained analysis problem solution process without any changes. In this case, even though only load variation bounds $\mathbf{F}_{inf}$, $\mathbf{F}_{sup}$ are given, the particular loading history $\mathbf{F}(t)$ is taken into account (the same as in one force $F(t)$ case).

If sequentially load variation field is extended in line with an arbitrary increment, not changing its form (that are quadrangles marked by dotted lines in Fig 2), elastic locus apexes $j \in J$ in the mathematical model (27)–(29) yield conditions (29). Then the analysis problem (27)–(29) obtains the form:

minimise

$$\frac{1}{2}\sum_k \left(\mathbf{S}_{r\Sigma,k} + \Delta\mathbf{S}_{rk}^{\nu}\right)^T [D_k]\left(\mathbf{S}_{r\Sigma,k} + \Delta\mathbf{S}_{rk}^{\nu}\right), \quad (41)$$

subject to

$$\sum_k [A_k]\Delta\mathbf{S}_{rk}^{\nu} = \mathbf{0}, \quad (42)$$

$$\Delta\mathbf{S}_{rk}^{\nu} = \left(\Delta\mathbf{S}_{rk1}^{\nu}, \Delta\mathbf{S}_{rk2}^{\nu}, \ldots, \Delta\mathbf{S}_{rkl}^{\nu}, \ldots, \Delta\mathbf{S}_{rk\eta_k}^{\nu}\right)^T,$$

$$\varphi_{kl,j} = C_k - f_{kl,j}\left(\mathbf{S}_{c,kl,j}^{\nu} + \Delta\mathbf{S}_{rkl}^{\nu}\right) \geq 0,$$

$$\mathbf{S}_{c,kl,j}^{\nu} = \mathbf{S}_{e,kl,j}^{\nu} + \mathbf{S}_{r\Sigma,kl} =$$

$$= \sum_{\nu} \Delta\mathbf{S}_{e,kl,j}^{\nu-1} + \Delta\mathbf{S}_{e,kl,j}^{\nu} + \sum_{\nu} \Delta\mathbf{S}_{r,kl}^{\nu-1},$$

$$C_k = (S_{0k})^2, \ k \in K, \ l \in L, \ j \in J. \quad (43)$$

Problem (41)–(43) optimal solution is vector of residual force increments $\Delta \mathbf{S}_{rk}^{*\nu}$ at the end of ν-th stage. At the end of loading process, when $\nu = z$, residual forces $\mathbf{S}_r^*$ are obtained from the formula (32): $\mathbf{S}_r^* = \sum_{\nu=1}^{z} \Delta \mathbf{S}_r^{*\nu}$. The Kuhn-Tucker conditions (33)–(34) remain analogic to conditions (33)–(34):

$$[D]\Delta \mathbf{S}_r^{*\nu} + [D]\mathbf{S}_{r\Sigma} + [\nabla\varphi]^T \boldsymbol{\lambda}^{*\nu} - [A]^T \mathbf{u}_r^{*\nu} = \mathbf{0},$$

$$\boldsymbol{\lambda}^{*\nu T}\boldsymbol{\varphi} = \mathbf{0}, \quad \boldsymbol{\lambda}^{*\nu} \geq \mathbf{0}, \quad \nu \in V.$$

Only plastic strains at the end of the ν-th stage are calculated taking into account all apexes $j \in J$ of elastic force locus:

$$\Theta_{pkl}^{*\nu} = \sum_j \left[\nabla\varphi_{kl,j}(\mathbf{S}_{c,kl,j}^{\nu} + \Delta \mathbf{S}_{rkl}^{*\nu})\right]^T \lambda_{kl,j}^{*\nu}, \qquad (44)$$

$$k \in K, \; l \in L, \; j \in J.$$

Residual displacements $\mathbf{u}_r$ are determined from formulas (36), (38), residual forces $\mathbf{S}_r$ can be checked applying the formula (39): $\mathbf{S}_r^* = [G]\,\Theta_p^*$.

Analysis of problem (41)–(43) optimal solution is actual residual forces $\mathbf{S}_r^*$; the result of iterative solution is actual residual displacements $\mathbf{u}_r^*$ and plastic strains Θ_p^* as well. Mathematical model (27)–(29) is a particular case of the problem (41)–(43), when $j = 1$.

4.3. Transformed mathematical model

The mathematical model (41)–(43) can be transformed applying the residual force influence matrix

$$[G] = [D]^{-1}[A]^T\left([A][D]^{-1}[A]^T\right)^{-1}[A][D]^{-1} - [D]^{-1}$$

and plastic strains $\Theta_{p\Sigma}$ at the beginning of the ν-th stage. Then residual forces $\mathbf{S}_{r\Sigma} = \sum_\nu \Delta \mathbf{S}_r^{\nu-1}$ are calculated according to the formula

$$\mathbf{S}_{r\Sigma} = [G]\,\Theta_p^{\nu-1}.$$

Object function expression (41) is rewritten as follows:

$$\frac{1}{2}\Theta_{p\Sigma}^T[G]^T[D][G]\Theta_{p\Sigma} +$$

$$+\frac{1}{2}\left(\Delta \mathbf{S}_r^{\nu}\right)^T[D]\Delta \mathbf{S}_r^{\nu} + \left(\Delta \mathbf{S}_r^{\nu}\right)^T[D][G]\Theta_{p\Sigma}. \qquad (45)$$

The first member of the expression (45) is constant

$$\frac{1}{2}\Theta_{p\Sigma}^T[G]^T[D][G]\Theta_{p\Sigma} = const \qquad (46)$$

and does not influence the optimal solution $\Delta \mathbf{S}_r^{*\nu}$ of analysis problem (41)–(43) determination. Therefore this constant member (46) is not necessary to be incorporated into the object function. Mathematical model of incremental structural analysis at shakedown (41)–(43) obtains the form:

$$\text{minimise} \; \left[\frac{1}{2}\Delta \mathbf{S}_r^{\nu T}[D]\Delta \mathbf{S}_r^{\nu} + \Delta \mathbf{S}_r^{\nu T}[D][G]\Theta_{p\Sigma}\right] \qquad (47)$$

subject to

$$[A]\Delta \mathbf{S}_r^{\nu} = \mathbf{0}, \qquad (48)$$

$$\Delta \mathbf{S}_r^{\nu} = \left(\Delta \mathbf{S}_{r1}^{\nu}, \Delta \mathbf{S}_{r2}^{\nu}, \ldots, \Delta \mathbf{S}_{rk}^{\nu}, \ldots, \Delta \mathbf{S}_{r\eta}^{\nu}\right)^T,$$

$$\Delta \mathbf{S}_{rk}^{\nu} = \left(\Delta \mathbf{S}_{rk1}^{\nu}, \Delta \mathbf{S}_{rk2}^{\nu}, \ldots, \Delta \mathbf{S}_{rkl}^{\nu}, \ldots, \Delta \mathbf{S}_{rk\eta_k}^{\nu}\right)^T,$$

$$\varphi_{kl,j} = C_k - f_{kl,j}\left(\mathbf{S}_{c,kl,j}^{\nu} + \Delta \mathbf{S}_{rkl}^{\nu}\right) \geq 0,$$

$$\mathbf{S}_{c,kl,j}^{\nu} = \mathbf{S}_{e,kl,j}^{\nu} + \mathbf{S}_{r\Sigma,kl} =$$

$$= \sum_\nu \Delta \mathbf{S}_{e,kl,j}^{\nu-1} + \Delta \mathbf{S}_{e,kl,j}^{\nu} + \sum_\nu \Delta \mathbf{S}_{r,kl}^{\nu-1},$$

$$C_k = (S_{0k})^2, \; k \in K, \; l \in L, \; j \in J. \qquad (49)$$

Optimal solution $\Delta \mathbf{S}_r^{*\nu}$ is achieved when the Rozen optimality criterion (Kuhn-Tucker conditions) [3] is satisfied:

$$[D]\Delta \mathbf{S}_r^{*\nu} + [D][G]\Theta_{p\Sigma} + [\nabla\varphi]^T\boldsymbol{\lambda}^{*\nu} - [A]^T\mathbf{u}_r^{*\nu} = \mathbf{0}, \qquad (50)$$

$$\boldsymbol{\lambda}^{*\nu T}\boldsymbol{\varphi} = \mathbf{0}, \quad \boldsymbol{\lambda}^{*\nu} \geq \mathbf{0}. \qquad (51)$$

Physical meaning of the second and third members of relations (50) is explained in more detail. Member $[D][G]\Theta_{p\Sigma}$ means elastic strains from accumulated plastic strains $\Theta_{p\Sigma} = \sum_\nu \Delta\Theta_p^{\nu-1}$ at the beginning of ν-th stage because

$$\mathbf{S}_r^{\nu-1} = [G]\Theta_{p\Sigma}. \qquad (52)$$

But then

$$[D][G]\Theta_{p\Sigma} = [D]\Big\{[D]^{-1}[A]^T\left([A][D]^{-1}[A]^T\right)^{-1} \times$$

$$\times [A][D]^{-1} - [D]^{-1}\Big\}\Theta_{p\Sigma} =$$

$$= \left\{[A]^T\left([A][D]^{-1}[A]^T\right)^{-1}[A][D]^{-1} - [I]\right\}\Theta_{p\Sigma} =$$

$$= [A]^T\mathbf{u}_{r\Sigma} - \Theta_{p\Sigma}. \qquad (53)$$

Influence matrix of residual displacements $[H] = \left([A][D]^{-1}[A]^T\right)^{-1}[A][D]^{-1}$ is applied to influence the expression (53). Then applying relation (53), geometric equations (50) can be written as follows:

$$[D]\Delta \mathbf{S}_r^{*\nu} + [A]^T \mathbf{u}_{r\Sigma} - \mathbf{\Theta}_{p\Sigma} + \mathbf{\Theta}_p^{*\nu} - [A]^T \mathbf{u}_r^{*\nu} = \mathbf{0}. \quad (54)$$

Having introduced vectors of residual displacement and plastic strain increments in the ν-th stage

$$\Delta \mathbf{u}_r^{*\nu} = \mathbf{u}_r^{*\nu} - \mathbf{u}_{r\Sigma}, \quad (55)$$

$$\Delta \Theta_p^{*\nu} = \Theta_p^{*\nu} - \Theta_{p\Sigma} \quad (56)$$

geometric equations (54) are rewritten by means of residual force, displacement and plastic strains increments as follows:

$$[D]\Delta \mathbf{S}_r^{*\nu} + \Delta \Theta_p^{*\nu} = [A]^T \Delta \mathbf{u}_r^{*\nu}. \quad (57)$$

Full equation system for the structure undergone a plastic deformation is next to consideration. It consists of analysis problem constraints and Kuhn-Tucker conditions (50)–(51):

$$[A]\Delta \mathbf{S}_r^{\nu} = \mathbf{0},$$

$$\varphi = \mathbf{C} - \mathbf{f}\left(\mathbf{S}_e^{\nu} + \Delta \mathbf{S}_r^{\nu}\right) \geq \mathbf{0}$$

$$(\varphi = (\varphi_{kl,j})^T \text{ for all } k \in K,\ l \in L,\ j \in J),$$

$$[D]\Delta \mathbf{S}_r^{\nu} + [D][G]\Theta_{p\Sigma} + [\nabla\varphi]^T \lambda^{\nu} - [A]^T \mathbf{u}_r^{\nu} = \mathbf{0},$$

$$\lambda^{\nu T}\varphi = \mathbf{0}, \quad \lambda^{\nu} \geq \mathbf{0}. \quad (58)$$

Having solved (58), formulas for residual force increment $\Delta \mathbf{S}_r^{\nu}$ at the ν-th stage and total residual displacements $\mathbf{u}_r^{\nu}$ at the end of stage calculation are obtained:

$$\Delta \mathbf{S}_r^{\nu} = [G]\left([D][G]\Theta_{p\Sigma} + [\nabla\varphi]^T \lambda^{\nu}\right) =$$

$$= \left\{[D]^{-1}[A]^T\left([A][D]^{-1}[A]^T\right)^{-1}[A][D]^{-1} - [D]^{-1}\right\}$$

$$\left([D][G]\Theta_{p\Sigma} + [\nabla\varphi]^T \lambda^{\nu}\right) =$$

$$= \left\{[D]^{-1}[A]^T\left([A][D]^{-1}[A]^T\right)^{-1}[A] - [I]\right\}[G]\Theta_{p\Sigma} +$$

$$+ [G][\nabla\varphi]^T \lambda^{\nu}\Big) = -[G]\Theta_{p\Sigma} + [G]\Theta_p^{\nu} = [G]\Delta\Theta_p^{\nu},$$

$$\mathbf{u}_r^{\nu} = [H]\left([D][G]\Theta_{p\Sigma} + [\nabla\varphi]^T \lambda^{\nu}\right) =$$

$$= \left([A][D]^{-1}[A]^T\right)^{-1}[A][D]^{-1}\left([D][G]\Theta_{p\Sigma} + [\nabla\varphi]^T \lambda^{\nu}\right) =$$

$$= [H][\nabla\varphi]^T \lambda^{\nu} = [H]\Theta_p^{\nu}. \quad (59)$$

4.4. About cross-section unloading phenomenon

According to the associative flow rule [6],

$$\lambda \geq 0, \text{ if } \varphi(\mathbf{S}) = 0 \quad (60)$$

and there is not any unloading phenomenon. Strictly speaking, unloading phenomenon is determined by the following condition:

$$\dot{\varphi}(\mathbf{S}) = \frac{\partial\varphi(\mathbf{S})}{\partial \mathbf{S}}\dot{\mathbf{S}} < 0. \quad (61)$$

Here $\dot{\mathbf{S}}$ is force velocity.

In the section 3.2 unloading phenomenon is determined as a case for an actual loading process, when

$$\lambda_i > 0, \quad (62)$$

$$\varphi_i = C_i - f_i(\mathbf{S}_i) \geq 0, \quad (63)$$

$$i = 1, 2, \ldots, \zeta, \quad i \in I.$$

Applying incremental method for structural analysis at shakedown it is possible to fix satisfaction of the condition (62)–(63). Mathematical models (27)–(29), (41)–(43) and (47)–(49) of analysis problem can serve for that. All these models of shakedown analysis are related to Haar-Kármán principle. This principle requires that during all loading stages the unloading phenomenon (when conditions (62)–(63) are satisfied) does not appear in any structure cross-section. When Mises yield conditions are applied, Haar-Kármán principle will be right if one more requirement is satisfied. That is the stress state, satisfying condition

$$\varphi(\mathbf{S}) = 0, \quad (64)$$

and it has to remain unchanged during the plastic deformation.

The above-mentioned requirements are taken into account verifying the sign of plastic multiplier increments at each calculation stage:

$$\Delta\lambda_i^{\nu} = \lambda_i^{\nu} - \lambda_i^{\nu-1}, \quad i = 1, 2, \ldots, \zeta, \quad i \in I. \quad (65)$$

When it is noticed that

$$\Delta\lambda_i^{\nu} < 0, \quad (66)$$

it means that the unloading phenomenon is developing in the i-th cross-section. Even though this moment is fixed formally, the analysis problem solution is continued in order to be sure that the unloading phenomenon really appeared in the cross-section, ie that conditions (62)–(63) are satisfied.

In the paper proposed mathematical models of incremental analysis allow the determination of unloading phenomenon only at one (the first) cross-section. Having fixed that, for further shakedown analysis the expression of yield condition must be modified (fictitious plasticity constant is introduced) in the mathematical models (27)–(29), (41)–(43) and (47)–(49). New problems are not detailed here, just creation of technique allowing to identify the moment of unloading phenomenon appearance at the cross-section.

5. Incremental shakedown analysis of plate

5.1. Plate analysis mathematical model

Mathematical model of the bending plate static formulation analysis is obtained from the problem (41)–(43):

$$\text{minimise } \frac{1}{2}\Delta\mathbf{M}_r^{\nu T}[D]\Delta\mathbf{M}_r^{\nu} + \Delta\mathbf{M}_r^{\nu T}[D][G]\Theta_{p\Sigma}, \quad (67)$$

subject to

$$[A]\Delta\mathbf{M}_r^{\nu} = \mathbf{0}, \quad (68)$$

$$\Delta\mathbf{M}_{rk}^{\nu} = \left(\Delta\mathbf{M}_{rk1}^{\nu}, \Delta\mathbf{M}_{rk2}^{\nu}, ..., \Delta\mathbf{M}_{rkl}^{\nu}, ..., \Delta\mathbf{M}_{rk\eta_k}^{\nu}\right)^T,$$

$$\varphi_{kl,j} = C_k - \mathbf{M}_{kl,j}^{\nu T}[\Phi]\mathbf{M}_{kl,j}^{\nu} \geq 0, \quad C_k = (M_{0k})^2,$$

$$\mathbf{M}_{kl,j}^{\nu} = \mathbf{M}_{c,kl,j}^{\nu} + \Delta\mathbf{M}_{r,kl}^{\nu},$$

$$\mathbf{M}_{c,kl,j}^{\nu} = \mathbf{M}_{e,kl,j}^{\nu} + \mathbf{M}_{r\Sigma,kl} =$$

$$= \sum_{\nu}\Delta\mathbf{M}_{e,kl,j}^{\nu-1} + \Delta\mathbf{M}_{e,kl,j}^{\nu} + \sum_{\nu}\Delta\mathbf{M}_{r,kl}^{\nu-1},$$

$$k \in K, \ l \in L, \ j \in J. \quad (69)$$

A particular case of mathematical model (67)–(69) is obtained for certain loading history $\mathbf{F}(t)$ consideration:

$$\text{minimise } \frac{1}{2}\Delta\mathbf{M}_r^{\nu T}[D]\Delta\mathbf{M}_r^{\nu} + \Delta\mathbf{M}_r^{\nu T}[D][G]\Theta_{p\Sigma}, \quad (70)$$

subject to

$$[A]\Delta\mathbf{M}_r^{\nu} = \mathbf{0}, \quad (71)$$

$$\Delta\mathbf{M}_{rk}^{\nu} = \left(\Delta\mathbf{M}_{rk1}^{\nu}, \Delta\mathbf{M}_{rk2}^{\nu}, ..., \Delta\mathbf{M}_{rkl}^{\nu}, ..., \Delta\mathbf{M}_{rk\eta_k}^{\nu}\right)^T,$$

$$\varphi_{kl} = C_k - \mathbf{M}_{kl}^{\nu T}[\Phi]\mathbf{M}_{kl}^{\nu} \geq 0, \quad C_k = (M_{0k})^2,$$

$$\mathbf{M}_{kl}^{\nu} = \mathbf{M}_{c,kl}^{\nu} + \Delta\mathbf{M}_{r,kl}^{\nu}, \quad \mathbf{M}_{c,kl}^{\nu} = \mathbf{M}_{e,kl}^{\nu} + \mathbf{M}_{r\Sigma,kl},$$

$$k \in K, \ l \in L. \quad (72)$$

Here $[\Phi]$ is von Mises yield condition matrix. Problems (67)–(69), (70)–(72) unknowns are residual moment increments $\Delta\mathbf{M}_r^{\nu}$. After optimal solution $\Delta\mathbf{M}_r^{*\nu}$ determination of each ν-th loading stage, the plastic deformations $\Theta_p^{*\nu}$ and residual displacements $\mathbf{u}_r^{*\nu}$ are found.

5.2. Initial admissible solution for analysis problem

Solving the analysis problem (10)–(14) or (67)–(69), (70)–(72) by the Rozen project gradient method, the determination of initial admissible point $\mathbf{x}$ is important. Global solution of load optimization or optimal design problems obtained when structure has reached limit state resulted by cyclic-plastic failure, can serve for that. Load optimization mathematical model for bending plate reads:

$$\text{maximise } \left\{\mathbf{T}_{sup}^T\mathbf{F}_{sup} + \mathbf{T}_{inf}^T\mathbf{F}_{inf}\right\}, \quad (73)$$

subject to

$$[A]\mathbf{M}_r = \mathbf{0}, \quad (74)$$

$$\varphi_{kl,j} = C_k - \left(\mathbf{M}_{ekl,j} + \mathbf{M}_{rkl}\right)^T[\Phi]\left(\mathbf{M}_{ekl,j} + \mathbf{M}_{rkl}\right) \geq 0,$$

$$C_k = (M_{0k})^2, \quad k \in K, \quad l \in L, \quad j \in J. \quad (75)$$

As a numerical example will be shown for circular plate, in this case von Mises yield condition reads:

$$M_\rho^2 - M_\rho M_\Theta + M_\Theta^2 \leq (M_0)^2. \quad (76)$$

Here

$$[\Phi] = \begin{bmatrix} 1 & -0,5 \\ -0,5 & 1 \end{bmatrix}.$$

Optimal solution of problem (73)–(75) means load variation bounds $\mathbf{F}_{inf}^*$, $\mathbf{F}_{sup}^*$ and residual moments $\mathbf{M}_r^*$. Vector $\mathbf{M}_r^*$ is initial admissible point $\mathbf{x}$ of analysis problem solved applying the Rozen algorithm.

Having changed object function (73) of the problem (73)–(75) to minimise $\frac{1}{2}\mathbf{M}_r^T[D]\mathbf{M}_r = a^*$, analysis problem (10)–(14) mathematical model for plates can be obtained. In this case the loading history is neglected.

5.3. Numeric examples of the plate analysis problem

Perfectly elastic-plastic annular plate with external radii R (Fig 6) is under consideration. The hinge-supported on external contour plate is subjected to uniformly distributed load q $(0 \leq q \leq q_{sup})$ and to the internal contour uniformly distributed moment M $(0 \leq M \leq M_{sup})$. The limit bending moment of the plate M_0, stiffness $\mathcal{K}$ are prescribed. Equilibrium finite element method is applied for plate discretisation [26].

The considered example is simple and serves only for illustration of proposed solution technique. Therefore the segment of annular plate is subdivided into three finite elements ($k = 1, 2, 3$). An element k $(k \in K)$ contains three nodal points $l = 1, 2, 3$ $(l \in L)$. Hence the vector of bending moments due to the applied cylindric co-ordinate system is: $\mathbf{M}_k = (M_{\rho,k1}, M_{\theta,k1}, M_{\rho,k2}, M_{\theta,k2}, M_{\rho,k3}, M_{\theta,k3})^T = (\mathbf{M}_{k1}, \mathbf{M}_{k2}, \mathbf{M}_{k3})^T$.

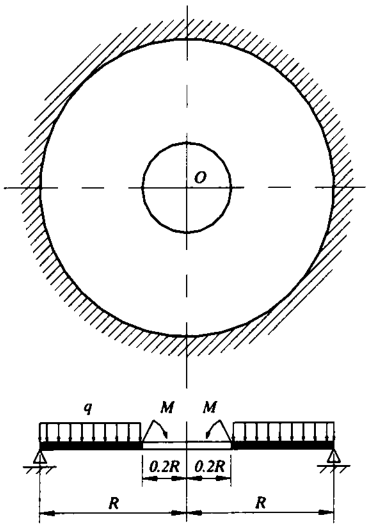

Fig 6. Annular plate

Here $\mathbf{M}_{kl} = \left(M_{\rho,kl},\ M_{\theta,kl}\right)^T$. Moment vector of plate discrete model is $\mathbf{M} = \left(\mathbf{M}_1, \mathbf{M}_2, ..., \mathbf{M}_k, ..., \mathbf{M}_\eta\right)^T$. There are 9 nodes where von Mises yield conditions are verified.

Non-linear problems are solved applying the software (Rozen algorithm, FORTRAN) created by the authors of this paper.

Example 1. Annular plate (Fig 6) is considered. Maximal variation bound q_{sup} $(0 \le q \le q_{sup})$ is to be found $(M = 0)$ according to cyclic-plastic failure conditions.

The problem is solved applying mathematical model (73)–(75):

$$\text{maximise } q = q_{sup}, \qquad (77)$$

subject to (74), (75).

Plate elastic moments $\mathbf{M}_e = \left(\mathbf{M}_{e1},\ \mathbf{M}_{e2},\ \mathbf{M}_{e3}\right)^T$ are linear q functions (Table 1) (one locus apex). Elastic moments, given in Table 1, are determined by exact formulas [27] and result in the influence matrix $[\alpha]$. The optimal solution of the problem (77)–(75) are $q^*_{sup} = 4{,}8950 M_0 R^{-2}$ and residual moments $\mathbf{M}^*_r$ (Table 2). Von Mises yield conditions (75) are satisfied as equalities in cross-sections 1, 3, 5, 6, 8, 9.

Table 1. Plate moments $\mathbf{M}_e(q)$

$M_{\rho e}$	$M_{\Theta e}$
0	0,36541
0,11003	0,24556
0,12492	0,20477
0,12492	0,20477
0,11224	0,17773
0,08496	0,15245
0,08496	0,15245
0,04703	0,12540
0	0,095257

Table 2. Plate moments $\mathbf{M}^*_r$

$M_{\rho r}$	$M_{\Theta r}$
0	– 0,78868
– 0,36012	– 0,84626
– 0,38891	0,09016
– 0,38891	0,07058
– 0,26685	0,24089
– 0,16325	0,35584
– 0,16325	0,34323
– 0,07639	0,45416
0	0,53370

Example 2. Annular plate subjected to the distributed load $q = 4{,}355047\ M_0 R^{-2}$ undergoes plastic strains not reaching cyclic-plastic failure $(M = 0)$. The analysis of problems (10)–(14) is performed.

Elastic moments $\mathbf{M}_e$ are obtained by multiplying moments presented in Table 1 by load $q = 4{,}355047 M_0 R^{-2}$ (Table 3). Problem (10)–(14) optimal solution is complementary deformation energy $a^* = 1{,}8647\ M_0^2 R^{-2} / \mathcal{K}$ and residual moments $\mathbf{M}^*_r$. Using mathematical-mechanical interpretation of the Rozen optimality criterion, plastic multipliers λ_{kl}, ie vector

$\lambda = (\lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{21}, \lambda_{22}, \lambda_{23}, \lambda_{31}, \lambda_{32}, \lambda_{33})^T$, vector of plastic strains $\mathbf{\Theta}_p$ and residual displacements $\mathbf{u}_r$ are determined (Table 3).

Example 3. Annular load is subjected to uniformly distributed load $q\,(0 \le q \le 4{,}355047\ M_0 R^{-2})$ and bending moment M $(0 \le M \le 0{,}176362\ M_0)$ (Fig 6). Loading program: when $M = 0$, the uniformly distributed load is increased by stages up to $q = 4{,}355047\ M_0 R^{-2}$, later, holding uniformly distributed load magnitude $q = 4{,}355047\ M_0 R^{-2}$ constant, moment M is increased by stages from 0 up to $M = 0{,}176362\ M_0$. Plate stress-strain state is considered in each loading stage. These results are compared to the analysis problem solution obtained neglecting the loading history.

Analysis problem is solved by two different ways: according to incremental analysis problem mathematical model (70)–(72) and according mathematical model (10)–(14) written for plates where loading history is neglected. Perfectly elastic plate moments, needed for analysis problem solution according both models, are written in Table 4.

Presented in Table 5 residual moments $\mathbf{M}^{*v}_r$ are obtained for all seven calculation stages of plate incremental analysis problem (70)–(72). Though optimal solution of each analysis problem stage is $\Delta\mathbf{M}^{*v}_r$, total residual moments at the end of v-th stage $\mathbf{M}^{*v}_r = \mathbf{M}_{r\Sigma} + \Delta\mathbf{M}^{*v}_r$ are presented in Table 5. The analysis problem (70)–(72) is solved considering in detail all loading history, but neglecting possible unloading phenomenon. For comparison, intermediate results of analysis problem (10)–(14) are presented in the same Table 5 obtained without loading history evaluation. Analysis of results in Table 5 shows that mathematical model of incremental analysis is formed correctly: the results of the 5th, 6th, 7th calculation stage coincide with residual moments $\mathbf{M}^*_r$ obtained via analysis problem (10)–(14) solution. If unloading phenomenon appeared during loading process, it is possible to determine analysing values of plastic multipliers λ^{*v} presented in Table 6. From the 5th stage when at the first node of the

Table 3. Optimal solution of annular plate analysis problem (10)–(14) (when $q = 4{,}355047\ M_0 R^{-2}$).

Elements	Nodes	$\mathbf{M}_e$	$\mathbf{M}_r^*$	$\mathbf{M}_e + \mathbf{M}_r^*$	λ	Θ_p
1	1	0	0	0	3,9100	–3,9099
		1,5914	– 0,59142	0,9996		7,8197
	2	0,47919	– 0,26157	0,21761	0	0
		1,0694	– 0,58375	0,48567		0
	3	0,54403	– 0,25774	0,28629	1,3546	–0,73059
		0,89178	0,22013	1,1119		2,6246
2	1	0,54403	– 0,25774	0,28629	0,22850	– 0,12323
		0,89178	0,22011	1,1119		0,44271
	2	0,48881	– 0,14188	0,34693	0	0
		0,77402	0,28646	1,0605		0
	3	0,37000	– 0,067366	0,30264	0	0
		0,66393	0,22879	0,89271		0
3	1	0,37000	– 0,067366	0,30264	0	0
		0,66393	0,20029	0,86422		0
	2	0,20481	– 0,026192	0,17862	0	0
		0,54612	0,19275	0,73887		0
	3	0	0	0	0	0
		0,41485	0,14026	0,55511		0

Table 4. Vectors $\mathbf{M}_{e1}$, $\mathbf{M}_{e2}$, $\mathbf{M}_{e3}$

$\mathbf{M}_{e1}(q)$		$\mathbf{M}_{e2}(M)$		$\mathbf{M}_{e3}(q, M)$	
M_ρ	M_Θ	M_ρ	M_Θ	M_ρ	M_Θ
0	0,36541	1,0	–1,03710	1,0 M	0,36541 q – 1,03710 M
0,11003	0,24556	0,33350	–0,40802	0,11003 q + 0,33350 M	0,24556 q – 0,40802 M
0,12492	0,20477	0,14967	–0,23108	0,12492 q + 0,14967 M	0,20477 q – 0,23108 M
0,12492	0,20477	0,14967	–0,23123	0,12492 q + 0,14967 M	0,20477 q – 0,23123 M
0,11224	0,17773	0,074078	–0,15664	0,11224 q + 0,074078 M	0,17773 q – 0,15664 M
0,08496	0,15245	0,035815	–0,11881	0,08496 q + 0,035815 M	0,15245 q – 0,11881 M
0,08496	0,15245	0,035815	–0,11885	0,08496 q + 0,035815 M	0,15245 q – 0,11885 M
0,047028	0,12540	0,013807	–0,09697	0,047028 q + 0,013807 M	0,12540 q – 0,09697 M
0	0,095257	0	–0,083241	0,095257 M	0,095257 q – 0,083241 M

Table 5. Annular plate analysis problem (example 3): residual moments $\mathbf{M}_r^*$.

Elements	Nodes	Incremental analysis problem (70)–(72): $\mathbf{M}_r^{*\nu}$ at the end of each stage $\nu = 1, 2, \ldots, 7$ $(\mathbf{M}_r^{*\nu} = \mathbf{M}_{r\Sigma} + \Delta\mathbf{M}_r^{*\nu})$							Analysis problem (10)–(14): optimal solution $\mathbf{M}_r^*$		
		1	2	3	4	5	6	7			
		$q = 2{,}740$ $M = 0{,}0$	$q = 3{,}276$ $M = 0{,}0$	$q = 3{,}815$ $M = 0{,}0$	$q = 4{,}355$ $M = 0{,}0$	$q = 4{,}355$ $M = 0{,}1$	$q = 4{,}355$ $M = 0{,}15$	$q = 4{,}355$ $M = 0{,}176$	$q = 4{,}355$ $M = 0{,}1$	$q = 4{,}355$ $M = 0{,}15$	$Q = {,}355$ $M = 0{,}176$
1	1	0,00000	0,00000	0,00000	0,00000	0,00000	0,00000	0,00000	0,00000	0,00000	0,00000
		–0,00119	–0,19698	–0,39422	–0,59142	–0,74880	–0,83038	–0,87423	–0,74883	–0,83039	–0,87439
	2	–0,00045	–0,07423	–0,16326	–0,26158	–0,33769	–0,37733	–0,39870	–0,33770	–0,37734	–0,39878
		–0,00071	–0,11710	–0,32254	–0,58376	–0,77811	–0,88000	–0,93515	–0,77814	–0,88004	–0,93535
	3	–0,00021	–0,03429	–0,12742	–0,25774	–0,35234	–0,40214	–0,42916	–0,35235	–0,40216	–0,42926
		0,00184	0,30530	0,34644	0,22012	0,16166	0,12791	0,10863	0,16167	0,12787	0,10859
2	1	–0,00021	–0,03429	–0,12742	–0,25774	–0,35234	–0,40214	–0,42916	–0,35235	–0,40216	–0,42926
		0,00029	0,03949	0,14679	0,22012	0,16163	0,12789	0,09219	0,16165	0,12786	0,09216
	2	–0,00009	–0,01724	–0,06404	–0,14190	–0,21951	–0,26459	–0,29145	–0,21951	–0,26181	–0,29153
		0,00032	0,04142	0,15381	0,28636	0,31517	0,29184	0,27761	0,31517	0,29181	0,27759
	3	–0,00003	–0,00822	–0,03060	–0,06741	–0,11470	–0,15483	–0,17624	–0,11472	–0,15486	–0,17631
		0,00018	0,01921	0,07103	0,22852	0,38462	0,37245	0,39553	0,38457	0,37241	0,39550
3	1	–0,00003	–0,00822	–0,03060	–0,06741	–0,11470	–0,15483	–0,17625	–0,11472	–0,15486	–0,17631
		0,00009	0,02445	0,09097	0,20041	0,34084	0,40647	0,38533	0,34124	0,40647	0,38447
	2	–0,00001	–0,00320	–0,01190	–0,02621	–0,04462	–0,06510	–0,08113	–0,04459	–0,06511	–0,08126
		0,00008	0,02353	0,08755	0,19287	0,32818	0,43811	0,49166	0,32824	0,43818	0,49176
	3	0,00000	0,00000	0,00000	0,00000	0,00000	0,00000	0,00000	0,00000	0,00000	0,00000
		0,00006	0,01712	0,06370	0,14035	0,23910	0,39584	0,55606	0,23861	0,39599	0,55765

Table 6. Annular plate analysis problem (example 3): plastic multipliers λ^* .

Elements	Nodes	Incremental analysis problem (70)–(72): λ^{*v} at the end of each stage v=1,2,...,7							Analysis problem (10)–(14): optimal solution λ^*		
		1	2	3	4	5	6	7			
		$q=2{,}740$ $M=0{,}0$	$q=3{,}276$ $M=0{,}0$	$q=3{,}815$ $M=0{,}0$	$q=4{,}355$ $M=0{,}0$	$q=4{,}355$ $M=0{,}1$	$q=4{,}355$ $M=0{,}15$	$q=4{,}355$ $M=0{,}176$	$q=4{,}355$ $M=0{,}1$	$q=4{,}355$ $M=0{,}15$	$q=4{,}355$ $M=0{,}176$
1	1	0,0035	0,5852	1,9089	3,9107	6,1804	8,2989	23,4420	6,1810	8,3010	23,5500
	2										
	3			0,4869	1,3549	2,2712	3,1349	9,5419	2,2713	3,1355	9,5870
2	1				0,2292	$\boxed{0{,}5051}$	0,3936		0,5049	0,3934	
	2					0,7796	2,4158	15,4320	0,7801	2,4175	15,5220
	3							0,4437			0,4488
3	1						0,2316			0,2319	
	2							10,9700			11,0510
	3										

Table 7. Annular plate analysis problem (example 3): plastic strains Θ_p^*.

Elements	Nodes	Incremental analysis problem (70)–(72): Θ_p^{*v} at the end of each stage v = 1, 2,...,7							Analysis problem (10)–(14): optimal solution Θ_p^*		
		1	2	3	4	5	6	7			
		$q=2{,}740$ $M=0{,}0$	$q=3{,}276$ $M=0{,}0$	$q=3{,}815$ $M=0{,}0$	$q=4{,}355$ $M=0{,}0$	$q=4{,}355$ $M=0{,}1$	$q=4{,}355$ $M=0{,}15$	$q=4{,}355$ $M=0{,}176$	$q=4{,}355$ $M=0{,}1$	$q=4{,}355$ $M=0{,}15$	$q=4{,}355$ $M=0{,}176$
1	1	–0,00352	–0,58524	–1,90880	–3,91050	–7,08460	–10,0960	–29,3680	–7,0851	–10,0980	–29,5040
		0,00704	1,17050	3,81760	7,82100	12,31500	16,45800	46,3330	12,3160	16,4610	46,5480
	2										
	3			–0,20905	–0,73076	–1,64240	–2,55630	–8,24640	–1,6425	–2,5569	–8,2866
				0,92821	2,62520	4,48880	6,23590	19,0250	4,4891	6,2372	19,1170
2	1				–0,12362	–3,65230	–0,32094		–0,36511	–0,32081	
					0,44407	0,99820	0,78288		0,99788	0,78256	
	2					–0,45299	–1,60200	–10,9670	–0,45329	–1,60320	–11,0330
						1,51840	4,748500	30,4670	1,519400	4,75170	30,6460
	3							–0,31299			–0,31667
								0,87548			0,88567
3	1						–0,15484			–0,15505	
							0,45542			0,456020	
	2							–8,92320			–8,98170
								21,82000			21,9800
	3										

second element maximal value of plastic multiplier is $\lambda_{21}^{*5} = 0{,}5051$ (Table 6, marked box), unloading starts, as $\Delta\lambda_{21}^6 = \lambda_{21}^6 - \lambda_{21}^5 = 0{,}3936 - 0{,}5051 = -0{,}1115 < 0$. It is obvious that the precision of solution depends on duration of loading stages v. Determination of that fact is the main result of the method proposed in this paper. In Table 6 presented magnitudes of plastic multipliers were obtained from analysis problem (10)–(14) solution for fixed q and M values (three last columns of Table 6). If problem (10)–(14) was solved only for that case, when $q=4{,}355$ and $M=0{,}15$, then it would not be possible to identify unloading phenomenon only according $\lambda_{21}^* = 0{,}3934$ magnitude. It once more confirms the necessity of incremental analysis when shakedown process is considered. But both models – (10)–(14) and (70)–(72) – do not simulate actual deformation process when unloading phenomenon appeared in the structure: the results of incremental analysis at the 6th and 7th stages are obtained neglecting Haar-Kármán principle. Solution results of mathematical model (70)–(72) correspond to system work without unloading phenomenon. Plastic strains Θ_p^{*v} (or Θ_p^*) of annular plate are calculated applying formula (18) and shown in Table 7, residual displacements $\mathbf{u}_r^{*v}$ (or $\mathbf{u}_r^*$) – in Table 8.

Table 8. Annular plate analysis problem (example 3): residual displacements $\mathbf{u}_r^{*\nu}$.

№	Incremental analysis problem (70)–(72): $\mathbf{u}_r^{*\nu}$ at the end of each stage $\nu=1,2,...,7$							Analysis problem (10)–(14): optimal solution $\mathbf{u}_r^{*}$		
	1	2	3	4	5	6	7			
	$q=2{,}740$ $M=0{,}0$	$q=3{,}276$ $M=0{,}0$	$q=3{,}815$ $M=0{,}0$	$q=4{,}355$ $M=0{,}0$	$q=4{,}355$ $M=0{,}1$	$q=4{,}355$ $M=0{,}15$	$q=4{,}355$ $M=0{,}176$	$q=4{,}355$ $M=0{,}1$	$q=4{,}355$ $M=0{,}15$	$q=4{,}355$ $M=0{,}176$
1	–0,00413	–0,68660	–2,19030	–4,55310	–8,26270	–11,9897	–39,0417	–8,25700	–11,9860	–39,2280
2	0,00194	0,32184	1,15497	2,50937	4,20407	5,92777	20,36977	4,20440	5,92910	20,4750
3	0,00098	0,16267	0,57841	1,28481	2,25936	3,32086	.13,00616	2,25960	3,32180	13,0770
4	0,00126	0,20905	0,77951	1,70917	2,84457	3,99087	13,72197	2,84470	3,99170	13,7930
5	–0,00216	–0,35951	–1,33580	–2,91870	–4,91290	–6,98520	–24,5112	–4,91330	–6,98680	–24,6380
6	0,00088	0,14544	0,54038	1,19492	2,02389	2,88750	10,5807	2,02410	2,88820	10,6370
7	0,00088	0,14557	0,54085	1,14535	1,97783	3,00623	11,85463	1,97810	3,00710	11,9180
8	0,00037	0,06161	0,22891	0,53743	0,89203	1,16711	3,88181	0,89207	1,16730	3,9026
9	–0,00159	–0,26355	–0,97923	–2,15473	–3,66743	–5,27783	–19,9378	–3,66780	–5,27920	–20,0440
10	0,00039	0,06507	0,24176	0,53197	0,90542	1,30641	4,89431	0,90550	1,30670	4,9206
11	0,00039	0,06504	0,24165	0,53172	0,90500	1,25884	5,82754	0,90508	1,25910	5,8607
12	–0,00001	–0,00123	–0,00456	–0,01003	–0,01707	0,01095	–0,80697	–0,01707	0,01099	–0,81265
13	–0,00139	–0,23050	–0,85642	–1,88452	–3,20752	–4,61752	–17,4695	–3,20770	4,61860	–17,5640

Physical meaning of residual displacement vector $\mathbf{u}_r$ components can be detected from dual relations between equilibrium and geometric equations. However, in the 6th and 7th stages Θ_p and $\mathbf{u}_r$ they correspond to system work without an unloading phenomenon.

Conclusions

Dual mathematical programming problems simulate the actual behaviour of structure at shakedown only when there is not any unloading phenomenon in the structure cross-sections. Rozen optimality criterion (Kuhn-Tucker conditions) ensures compatibility of residual strains and allows plastic strain and residual displacement increment calculation without dual problem solution. Only due to incremental analysis it is possible to *fixe* the appearance of unloading phenomenon at structure cross-section during shakedown. The proposed technique allows the determination of unloading phenomenon just in one (the first) structure cross-section and it cannot be applied for simulating the actual plastic deformation.

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