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NON-LINEAR RESPONSE FUNCTIONS FOR TRANSVERSELY ISOTROPIC ELASTIC MEMBRANES

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1. Introduction

The non-linear theory of flexible membranes has long been used to model diverse phenomena ranging from pneumatic and tension structures in civil engineering to biomembranes in medical sciences (see [1, 2] and references cited therein). One of the main difficulties in applying the general theory of membranes to reasonable physical situations for which it is intended, is the need for explicit representations of the constitutive response functions.

The general constitutive equation of an elastic membrane relates the tangential surface stress tensor (of the second Piola-Kirchhoff type) $\mathbf{S}(x,t)$ to the tangential surface deformation gradient $\mathbf{F}(x,t)$. Here $x \in M$ denotes a place occupied by a typical membrane particle in the chosen reference configuration, which is a smooth geometric surface M in the physical space [3, 4] as the notation and basic definitions are concerned). Any constitutive equation is further restricted by the principle of material frame-indifference and possible material symmetries. The frame-indifferent constitutive equations takes the form $\mathbf{S} = \widetilde{\mathbf{S}}(\mathbf{C})$ where the tangential surface deformation tensor (right Cauchy-Green type) $\mathbf{C}(x,t)$ is defined by $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ and $\widetilde{\mathbf{S}}$ is a given tensor function defining mechanical response of an elastic membrane [3, 5]. Further simplifications of such a constitutive relation are possible if a membrane exhibits certain symmetries in its response. In particular, if the membrane response is an isotropic relative to the undistorted reference configuration, then the representation theorems for isotropic functions in two dimensions applied to $\widetilde{\mathbf{S}}(\mathbf{C})$ yields the following constitutive equation [5]

$$\mathbf{S} = \tau_0(i_1,i_2)\mathbf{1}_0 + \tau_1(i_1,i_2)\mathbf{C}, \quad i_1 = \operatorname{tr}\mathbf{C}, \quad i_2 = \det\mathbf{C}, \tag{1}$$

where the response material functions

$$\tau_\Gamma = \tau_\Gamma(i_1,i_2), \quad \Gamma = 0,1, \tag{2}$$

are scalar functions of the principal invariants i_1 and i_2 of the deformation tensor $\mathbf{C}$. Physically, the constitutive relation (1) says that the stress in an isotropic elastic membrane does not depend on an arbitrary rotation of a local natural state. In this special case, the problem of the constitutive equations is reduced to the determination of two-scalar functions (2), which completely specify the mechanical properties of the membrane.

If a membrane is considered as a three-dimensional, thin shell-like body and if a constitutive equation for a material of such a body is known, then the two-dimensional constitutive equation (1) can be derived with the use of certain simplifying assumptions. Such an approach has long been used to derive the explicit form of the response functions (2) for membranes made of isotropic hyperelastic materials (eg [1, 6, 7]).

In this paper the explicit form of the response functions is derived for membranes made of transversely isotropic elastic materials under the assumption that fibre directions (preferred direction of anisotropy) coincide with normal to the reference configuration M of the membrane. In particular, it is shown that the two-dimensional response of the membrane is isotropic in this case. It is obvious, however, that the response functions (2) are different from their forms for isotropic materials.

2. Membranes made of transversely isotropic elastic materials

The mechanical response of a hyperelastic material is completely determined by the strain energy function $W = W(\mathbf{F})$, where $\mathbf{F}$ denotes the deformation gradient relative to the chosen reference configuration B of the body [8, 9]. In general, the possible forms of $W(\mathbf{F})$ are restricted by the frame-indifference principle and possible material symmetries. The principle of frame-indifference may be satisfied identically if the strain energy function is written as a given function $W = \widetilde{W}(\mathbf{C})$ of the right Cauchy-Green deformation tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. Then, the particular material symmetry is defined by the following condition [8, 9]

$$\widetilde{W}(\mathbf{C}) = \widetilde{W}(\mathbf{HCH}^T), \tag{3}$$

for all tensors $\mathbf{H} \in G$. Here G denotes the material symmetry group, the subgroup of the special linear group $SL(E)$ (often called unimodular group). In particular, a transversely isotropic hyperelastic material is defined by the condition (3) for all orthogonal tensors $\mathbf{H}(\boldsymbol{x}) \in O(E)$ such that $\mathbf{He}_0 = \mathbf{e}_0$. Here $\mathbf{e}_0(\boldsymbol{x})$ is a unit vector defining the axis of transverse isotropy of the material [8, 9]. This requirement implies that $\widetilde{W}(\mathbf{C})$ may be written as a function of the right Cauchy-Green deformation tensor $\mathbf{C}$ and of the unit vector $\mathbf{e}_0$, $W = \widetilde{W}(\mathbf{C}, \mathbf{e}_0)$. Moreover, the assumption that the direction of $\mathbf{e}_0$ has no mechanical significance implies that $\widetilde{W}(\mathbf{C}, \mathbf{e}_0)$ is an even function of $\mathbf{e}_0$. This condition is satisfied if the strain energy is written in the form $W = \widetilde{W}(\mathbf{C}, \mathbf{A}_0)$, where $\mathbf{A}_0 = \mathbf{e}_0 \otimes \mathbf{e}_0$ is often called the fabric tensor. Then, the invariance requirement (3) implies that $\widetilde{W}(\mathbf{C}, \mathbf{A}_0)$ is an isotropic function in both arguments. By the representation theorem for isotropic scalar-valued functions of two tensor arguments, the strain energy function takes the form [8, 9]

$$W = \widetilde{W}(I_1, I_2, I_3, I_4, I_5), \tag{4}$$

where I_A, $A = 1,2,3,4,5$, are joint invariants of $\mathbf{C}$ and $\mathbf{A}_0$ defined by

$$\begin{aligned} &I_1 = \operatorname{tr}\mathbf{C}, \; I_2 = \frac{1}{2}\left\{(\operatorname{tr}\mathbf{C})^2 - \operatorname{tr}\mathbf{C}^2\right\}, \; I_3 = \det\mathbf{C}, \\ &I_4 = \mathbf{e}_0 \bullet \mathbf{C}\mathbf{e}_0, \; I_5 = \mathbf{e}_0 \bullet \mathbf{C}^2\mathbf{e}_0. \end{aligned} \tag{5}$$

Then, the general form of three-dimensional constitutive equations may be derived from (4) with the use of known formulae for the derivatives of the invariant (5) with respect to the deformation tensor $\mathbf{C}$ [8, 9].

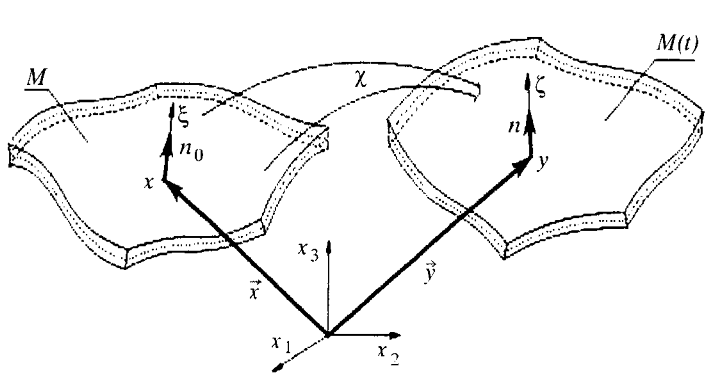
In this paper we shall be concerned with flexible membranes considered as a three-dimensional, thin shell-like bodies. The reference configuration of such a body may be described in the standard way [1, 6, 7]. Let $M \subset B$ be the mid-surface in the reference configuration of a smooth membrane. Then the position vector of any point $\boldsymbol{x} \in B$ may be expressed in the form

$$\begin{aligned} &\vec{\mathbf{x}}(\boldsymbol{x}, \xi) = \vec{x}(\boldsymbol{x}) + \xi \boldsymbol{n}_0(\boldsymbol{x}), \\ &\xi \in [-h_0/2, +h_0/2]. \end{aligned} \tag{6}$$

Here $\vec{x}(\boldsymbol{x})$ denotes the position vector of the corresponding point $\boldsymbol{x} \in M$ at the mid-surface and $\boldsymbol{n}_0(\boldsymbol{x})$ is the unit normal vector to M at the same point. Moreover, $h_0(\boldsymbol{x}) > 0$ denotes the initial (not necessarily uniform) thickness of the membrane.

In the same manner, the position vector of any point $\mathbf{y} \in B(t)$ in the current configuration $B(t)$ may be expressed in the form [8]

$$\begin{aligned} &\vec{\mathbf{y}}(\boldsymbol{y}, \zeta, t) = \vec{y}(\boldsymbol{y}, t) + \zeta \boldsymbol{n}(\boldsymbol{y}, t), \\ &\zeta \in [-h(\boldsymbol{y}, t)/2, +h(\boldsymbol{y}, t)/2], \end{aligned} \tag{7}$$



Deformation of a thin membrane

where $\bar{y}(y,t)$ denotes the position vector of the corresponding point $y \in M(t)$ at the mid-surface $M(t) \subset B(t)$. Moreover, $\boldsymbol{n}(y,t)$ is the unit normal vector to $M(t)$ and $h(y,t) > 0$ denotes the current thickness of the membrane.

This description involves the assumption according to which the three-dimensional deformation is such that the mid-surface in the reference configuration deforms into the mid-surface in the current configuration of the membrane. This is a consistent assumption within the theory of thin-membranes [1, 8] according to which $\varepsilon << 1$, where the small parameter $\varepsilon > 0$ is defined as the maximum of the ratio h_0 / R_0. Here R_0 denotes the smallest of the principal radii of curvature of M. Moreover, within the same error the normal coordinate ς in the current configuration of the membrane may be assumed in the form [1, 3, 7].

$$\zeta(y,\xi,t) = \lambda_\xi(y,t)\xi, \tag{8}$$

where through-the-thickness stretch $\lambda_\xi(\boldsymbol{x},t)$ determines the ratio of the current and initial thickness of the membrane, $h = \lambda_\xi h_0$. Then, the position vector (7) may be rewritten in the form

$$\begin{aligned}&\bar{\mathbf{y}}(y,\xi,t) = \bar{y}(y,t) + \lambda_\xi(y,t)\xi\boldsymbol{n}(y,t),\\&\xi \in [-h_0/2, +h_0/2].\end{aligned} \tag{9}$$

The assumption (8) implies that the transverse normal deformation is constant through-the-thickness of the membrane. This assumption has a number of important implications.

If the membrane is made up of a hyperelastic material whose mechanical response is determined by the 3D strain energy function $W = \widetilde{W}(\mathbf{C})$, then the two-dimensional strain energy function Φ (measured per unit area of the mid-surface M) may formally be defined as the integral of $\widetilde{W}(\mathbf{C})$ through the thickness of the membrane:

$$\begin{aligned}&\Phi = \int_{-h_0/2}^{+h_0/2} \widetilde{W}(\mathbf{C})\mu d\xi = h_0 \left\langle \widetilde{W}(\mathbf{C})\mu \right\rangle,\\&\langle\cdot\rangle \equiv \int_{-1/2}^{+1/2} (\cdot) d\hat{\xi}.\end{aligned} \tag{10}$$

Here $\hat{\xi} = \xi / h_0$ is the normalized coordinate in thickness direction and $\mu = 1 - 2H\xi + K\xi^2$, where H and K denotes the mean and Gaussian curvature of the mid-surface M. For thin membranes it may be assumed $\mu \cong 1$.

3. Three-dimensional deformation of the membranes

In continuum mechanics, the deformation gradient $\mathbf{F} = \nabla\bar{\mathbf{y}}$ and associated tensors are the fundamental quantities for the analysis of the local properties of the deformation. The most important implication of the assumptions underlying the theory of thin membranes is that $\mathbf{F}$ is constant through-the-thickness, ie $\mathbf{F} = \mathbf{F}(\boldsymbol{x},t)$. Moreover, taking into account the relation between the gradient operator ∇ in three-dimensional Euclidean space and the surface gradient operator ∇ [3], it follows from (9) that

$$\mathbf{F}(\boldsymbol{x},t) = \boldsymbol{F}(\boldsymbol{x},t)\boldsymbol{P}_0(\boldsymbol{x}) + \lambda_\xi(\boldsymbol{x},t)\boldsymbol{n}(\boldsymbol{x},t) \otimes \boldsymbol{n}_0(\boldsymbol{x}) . \tag{11}$$

Here $\boldsymbol{P}_0(\boldsymbol{x})$ denotes the canonical projection operator in the reference configuration of the membrane and $\boldsymbol{F}(\boldsymbol{x},t) = \nabla\bar{y}(\boldsymbol{x},t)$ denotes the surface deformation gradient, the linear map of the tangent space $T_x M$ into the three-dimensional Euclidean vector space $E \cong T_x\mathcal{E}$ (see [3, 4]). Actually, the codomain of $\boldsymbol{F}(\boldsymbol{x},t)$ is the tangent space $T_y M(t)$ and hence it may be expressed in the form

$$\boldsymbol{F}(\boldsymbol{x},t) = \nabla\bar{y}(\boldsymbol{x},t) = \boldsymbol{I}(y,t)\mathsf{F}(\boldsymbol{x},t) \tag{12}$$

where $\mathsf{F}(\boldsymbol{x},t)$ denotes the tangential deformation gradient, ie the linear map of $T_x M$ into $T_y M(t)$. In view of (12), the 3D deformation gradient (11) may be written in the form

$$\mathbf{F} = \boldsymbol{I}\mathsf{F}\boldsymbol{P}_0 + \lambda_\xi \boldsymbol{n} \otimes \boldsymbol{n}_0 . \tag{13}$$

It then follows that the tangential deformation gradient $\mathsf{F}(\boldsymbol{x},t)$ and through-the-thickness stretch $\lambda_\xi(\boldsymbol{x},t)$ completely described the local deformation of the membrane as a thin 3D body. There are a number of important implications of this fact.

Local deformation of a three-dimensional body can be decomposed into pure strains followed by the rigid rotation [8, 9]. This decomposition follows from the polar decomposition of the deformation gradient $\mathbf{F} = \mathbf{RU}$, where $\mathbf{U}$ is a symmetric, positive definite tensor (called the right stretch tensor) and $\mathbf{R}$ is a proper orthogonal tensor (called the rotation tensor). Similarly, the polar decomposition theorem applied to the tangential deformation gradient F yields $\mathsf{F} = \mathsf{RU}$, were $\mathsf{U}(\boldsymbol{x},t) : T_x M \to T_x M$ are two-dimensional symmetric, positive definite tensors and $\mathsf{R}(\boldsymbol{x},t) : T_x M \to T_y M(t)$ is a proper orthogonal tensor. In view of (13) we thus obtain

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \boldsymbol{I}\mathbf{R}\mathbf{U}\boldsymbol{P}_0 + \lambda_\xi \boldsymbol{n} \otimes \boldsymbol{n}_0 . \quad (14)$$

It then follows that 3D stretch tensor $\mathbf{U}$ and the 3D rotation tensor $\mathbf{R}$ are given by

$$\mathbf{U} = \boldsymbol{I}_0 \mathbf{U} \boldsymbol{P}_0 + \lambda_\xi \boldsymbol{n}_0 \otimes \boldsymbol{n}_0 , \ \mathbf{R} = \boldsymbol{I}\mathbf{R}\boldsymbol{P}_0 + \boldsymbol{n} \otimes \boldsymbol{n}_0 . \quad (15)$$

There are immediate and important implications of (15). In particular, the 3D Cauchy-Green deformation tensor is obtained in the form

$$\mathbf{C} = \boldsymbol{I}_0 \mathbf{C} \boldsymbol{P}_0 + \lambda_\xi^2 \boldsymbol{n}_0 \otimes \boldsymbol{n}_0 , \quad (16)$$

where the in-surface deformation tensors are defined by

$$\mathbf{C} = \boldsymbol{F}^T \boldsymbol{F} = \mathbf{F}^T \mathbf{F} = \mathbf{U}^2 . \quad (17)$$

This may be show by direct calculation. In fact, with the use of (14) we have

$$\begin{aligned} \mathbf{F}^T\mathbf{F} &= (\boldsymbol{I}_0 \boldsymbol{F}^T + \lambda_\xi \boldsymbol{n}_0 \otimes \boldsymbol{n})(\boldsymbol{F}\boldsymbol{P}_0 + \lambda_\xi \boldsymbol{n} \otimes \boldsymbol{n}_0) = \\ &= \boldsymbol{I}_0 \boldsymbol{F}^T \boldsymbol{F} \boldsymbol{P}_0 + \lambda_\xi (\boldsymbol{I}_0 \boldsymbol{F}^T)(\boldsymbol{n} \otimes \boldsymbol{n}_0) + \\ &+ \lambda_\xi (\boldsymbol{n}_0 \otimes \boldsymbol{n}) \boldsymbol{F} \boldsymbol{P}_0 + \lambda_\xi^2 (\boldsymbol{n}_0 \otimes \boldsymbol{n})(\boldsymbol{n} \otimes \boldsymbol{n}_0). \end{aligned} \quad (18)$$

Moreover, from the basic properties of the inclusion and projection operators it follows that

$$\begin{aligned} &(\boldsymbol{I}_0 \boldsymbol{F}^T)(\boldsymbol{n} \otimes \boldsymbol{n}_0) = \boldsymbol{I}_0 \boldsymbol{F}^T \boldsymbol{n} \otimes \boldsymbol{n}_0 = \mathbf{0}, \\ &(\boldsymbol{n}_0 \otimes \boldsymbol{n}) \boldsymbol{F} \boldsymbol{P}_0 = \boldsymbol{n}_0 \otimes \boldsymbol{I}_0 \boldsymbol{F}^T \boldsymbol{n} = \mathbf{0}, \\ &(\boldsymbol{n}_0 \otimes \boldsymbol{n})(\boldsymbol{n} \otimes \boldsymbol{n}_0) = \boldsymbol{n}_0 \otimes \boldsymbol{n}_0 , \end{aligned} \quad (19)$$

where the following identity has been used

$$\boldsymbol{I}_0 \boldsymbol{F}^T \boldsymbol{n} = \boldsymbol{I}_0 (\boldsymbol{I}\mathbf{F})^T \boldsymbol{n} = \boldsymbol{I}_0 \mathbf{F}^T \boldsymbol{P} \boldsymbol{n} = \mathbf{0} . \quad (20)$$

From (19) and (18) the formula (16) follows. Moreover, exactly the same calculations make it possible to derive many other relations between the 3D deformation tensors and their 2D counterparts for membranes. In fact, the relation (16) is a special case of the general formulae

$$\mathbf{C}^m = \boldsymbol{I}_0 \mathbf{C}^m \boldsymbol{P}_0 + \lambda_\xi^{2m} \boldsymbol{n}_0 \otimes \boldsymbol{n}_0 \quad (21)$$

for every integer $m = 0, \pm 1, \pm 2, \ldots$. Moreover, one may use (15) to show that

$$\mathbf{U}^m = \boldsymbol{I}_0 \mathbf{U}^m \boldsymbol{P}_0 + \lambda_\xi^{2m} \boldsymbol{n}_0 \otimes \boldsymbol{n}_0 , \quad (22)$$

for every integer $m = 0, \pm 1, \pm 2, \ldots$.

4. General form of 2D constitutive equations for membranes

By virtue of (16), the 3D strain energy function $\widetilde{W}(\mathbf{C})$ depends on the 3D Cauchy-Green deformation tensor $\mathbf{C}$ through the tangential deformation tensor $\mathbf{C}$ and through-the-thickness stretch λ_ξ only, $\widetilde{W}(\mathbf{C}) = \widetilde{W}(\mathbf{C}, \lambda_\xi)$. Hence, the 2D strain energy function Φ defined by (10) may be rewritten as

$$\Phi = h_0 \left\langle \widetilde{W}(\mathbf{C}, \lambda_\xi) \right\rangle . \quad (23)$$

It is seen now that the problem of determining the elastic response of the membrane reduces to the problem of expressing through-the-thickness stretch λ_ξ as a function of the deformation tensor $\mathbf{C}$. This may be achieved in at least two ways. For the time being let us assume that such a function has been found so we may write

$$\lambda_\xi(\boldsymbol{x}, t) = \Lambda(\mathbf{C}(\boldsymbol{x}, t)) \quad (24)$$

in which case the 2D strain energy function (23) becomes a function of the tangential deformation tensor $\mathbf{C}$ only:

$$\Phi = \widetilde{\Phi}(\mathbf{C}) = h_0 \left\langle \widetilde{W}(\mathbf{C}, \Lambda(\mathbf{C}) \right\rangle . \quad (25)$$

Standard argument may next be used to show that the constitutive equation of a membrane is determined by the relation [1, 5]

$$\mathbf{S} = \widetilde{\mathbf{S}}(\mathbf{C}) = 2\partial_{\mathbf{C}} \widetilde{\Phi}(\mathbf{C}) , \quad (26)$$

where $\mathbf{S}(\boldsymbol{x}, t)$ denotes the tangential surface stress tensor of the second Piola-Kirchhoff type.

Let us consider now the membrane made of a transversely isotropic material whose axis of isotropy coincides with the unit normal vector $\boldsymbol{n}_0$. Thus, that the 3D strain energy function satisfies the condition (3) for every orthogonal tensor $\mathbf{H} \in O(E)$ such that $\mathbf{H}\boldsymbol{n}_0 = \boldsymbol{n}_0$. With the use of (16) we have

$$\mathbf{H}\mathbf{C}\mathbf{H}^T = \mathbf{H}\boldsymbol{I}_0 \mathbf{C} \boldsymbol{P}_0 \mathbf{H}^T + \lambda_\xi^2 (\mathbf{H}\boldsymbol{n}_0 \otimes \mathbf{H}\boldsymbol{n}_0) , \quad (27)$$

where the following tensor identity has been used

$$\mathbf{H}(\boldsymbol{n}_0 \otimes \boldsymbol{n}_0)\mathbf{H}^T = \mathbf{H}\boldsymbol{n}_0 \otimes \mathbf{H}\boldsymbol{n}_0 . \quad (28)$$

Moreover, the tensor $\mathbf{H}(\boldsymbol{x})$ defined by $\mathbf{H} = \boldsymbol{P}_0 \mathbf{H} \boldsymbol{I}_0$ is necessarily orthogonal on the tangent space $T_x M$ and since $\mathbf{H}\boldsymbol{n}_0 = \boldsymbol{n}_0$, from (27) we have

$$\mathbf{H}\mathbf{C}\mathbf{H}^T = \mathbf{H}\boldsymbol{I}_0 \mathbf{C} \boldsymbol{P}_0 \mathbf{H}^T = \boldsymbol{I}_0 \mathbf{H}\mathbf{C}\mathbf{H}^T \boldsymbol{P}_0 . \quad (29)$$

It is seen now that if the 3D strain energy function satisfies the condition (3) for every orthogonal tensor $\mathbf{H} \in O(E)$, then the 2D strain energy function defined by (25) obey the following condition

$$\widetilde{\Phi}(\mathbf{C}) = \widetilde{\Phi}(\mathbf{HCH}^T), \qquad (30)$$

for all orthogonal tensors $\mathbf{H}(x) \in O(T_x M)$ provided that

$$\Lambda(\mathbf{C}) = \Lambda(\mathbf{HCH}^T). \qquad (31)$$

The physical sense of this result is actually self-evident. The conditions (3) defines a transversely isotropic material while the condition (30) is the definition of an isotropic response of the membrane [5]. Thus we have shown that if a membrane is made up of a transversely isotropic material whose fiber direction in the reference configuration coincide with the unit normal vector $\boldsymbol{n}_0(\boldsymbol{x})$, then the 2D mechanical response of the membrane is isotropic if and only if the condition (31) holds.

With the use of (16) the first three 3D invariants in the list (5) are obtained in the form

$$I_1 = \operatorname{tr}\mathbf{C} + \lambda_\xi^2, \; I_2 = \lambda_\xi^2 \operatorname{tr}\mathbf{C} + \det\mathbf{C}, \\ I_3 = \lambda_\xi^2 \det\mathbf{C}. \qquad (32)$$

Moreover, for a membrane made of a transversely isotropic material whose fiber direction in the reference configuration coincide with the unit normal vector, ie $\mathbf{e}_0(\mathbf{x}) = \boldsymbol{n}_0(\boldsymbol{x})$ for all points $\boldsymbol{x} \in M$, then from (21) we have

$$\boldsymbol{n}_0 \mathbf{C}^m \boldsymbol{n}_0 = \boldsymbol{n}_0 \bullet \boldsymbol{I}_0 \mathbf{C}^m \boldsymbol{P}_0 \boldsymbol{n}_0 + \lambda_\xi^{2m} \boldsymbol{n}_0 \bullet (\boldsymbol{n}_0 \otimes \boldsymbol{n}_0) \boldsymbol{n}_0, \qquad (33)$$

for every integer $m = \pm1, \pm2, \ldots$. With the use of the obvious relations and noting that $\boldsymbol{P}_0 \boldsymbol{n}_0 = \mathbf{0}$, we see that

$$\boldsymbol{n}_0 \bullet \mathbf{C}^m \boldsymbol{n}_0 = \lambda_\xi^{2m}, \quad m = \pm1, \pm2, \ldots. \qquad (34)$$

Denoting the principal invariants of the tangential deformation tensor $\mathbf{C}(\boldsymbol{x}, t)$ by

$$i_1 = \operatorname{tr}\mathbf{C}, \; i_2 = \det\mathbf{C} = \frac{1}{2}\left\{(\operatorname{tr}\mathbf{C})^2 - \operatorname{tr}\mathbf{C}^2\right\} \qquad (35)$$

it follows from (32) and (34) that the 3D principal invariants (5) and the 2D principal invariants (35) are related by

$$I_1 = i_1 + \lambda_\xi^2, \; I_2 = i_1 \lambda_\xi^2 + i_2, \; I_3 = i_2 \lambda_\xi^2, \\ I_4 = \lambda_\xi^2, \; I_5 = \lambda_\xi^4. \qquad (36)$$

Thus they are uniquely determined by the in-surface principal invariants (27) and the through-the-thickness stretch $\lambda_\xi(\boldsymbol{x}, t)$. Noting that $\mathbf{C}(\boldsymbol{x}, t)$ is a positive-definite tensor, it is seen i_1 and i_2 satisfies the condition

$$i_1^2 - 4 i_2 \geq 0, \qquad (37)$$

which together with $\lambda_\xi \geq 0$ define the natural domain of the response material functions.

If a membrane is made up of a transversely hyperelastic isotropic material whose fiber direction coincide with the unit normal vector in the reference configuration, then the relations (36) show that $W(I_1, \ldots, I_5)$ may be considered as a function of (i_1, i_2, λ_ξ). It is seen now that if λ_ξ can be uniquely determined by the in-surface principal invariants i_1 and i_2. As a result, whenever 3D strain energy is known for any transversely isotropic material, then

$$\Phi(i_1, i_2) = h_0 \langle W(I_1, I_2, I_3, I_4, I_5) \rangle. \qquad (38)$$

where the principal invariants $I_1, \ldots, I_5$ are uniquely determined in terms of i_1 and i_2.

In the case of hyperelastic isotropic membrane, the strain energy function $\Phi = \Phi(\mathbf{C})$ depends on $\mathbf{C}$ only through the principal invariants i_1 and i_2, ie $\Phi = \Phi(i_1, i_2)$. If this function is differentiable with respect to its arguments, then

$$\partial_{\mathbf{C}} \Phi = \Phi_1 (\partial_{\mathbf{C}} i_1) + \Phi_2 (\partial_{\mathbf{C}} i_2), \qquad (39)$$

where Φ_Λ, $\Lambda = 1, 2$, are defined by

$$\Phi_1(i_1, i_2) \equiv \partial_{i_1} \Phi(i_1, i_2), \\ \Phi_2(i_1, i_2) \equiv \partial_{i_2} \Phi(i_1, i_2). \qquad (40)$$

Keeping in mind that

$$\partial_{\mathbf{C}} i_1 = \mathbf{1}_0, \; \partial_{\mathbf{C}} i_2 = i_1 \mathbf{1}_0 - \mathbf{C}, \qquad (41)$$

the constitutive equation of an isotropic hyperelastic membrane takes the form

$$\mathbf{S} = 2\{(\Phi_1 + i_1 \Phi_2) \mathbf{1}_0 - \Phi_2 \mathbf{C}\}. \qquad (42)$$

Comparison of (1) and (42) yields the response functions (2) in terms of derivatives (40) of the strain energy function through the following relations

$$\tau_0(i_1, i_2) = 2\{\Phi_1(i_1, i_2) + i_1 \Phi_2(i_1, i_2)\}, \\ \tau_1(i_1, i_2) = -2\Phi_2(i_1, i_2). \qquad (43)$$

In this case the determination of the response functions τ_Γ, $\Gamma = 0, 1$, reduces to the determination of $\Phi_\Lambda = (i_1, i_2)$, $\Lambda = 1, 2$. If a material is homogenous in the normal direction to the mid-surface, then from (38) we may derive the following general formulae

$$\Phi_\Lambda = h_0 \sum_{K=1}^{5} W_K I_\Lambda^K, \; W_K \equiv \partial W / \partial I_K, \quad I_\Lambda^K \equiv \partial I_K / \partial i_\Lambda . \tag{44}$$

Substituting next (44) into (43) we finally obtain

$$\tau_0 = 2h_0 \sum_{A=1}^{5} (I_1^A + i_1 I_2^A) W_A, \quad \tau_1 = -2h_0 \sum_{A=1}^{5} I_2^A W_A . \tag{45}$$

There remain only to determine the relation (24).

5. Constrained transversely isotropic materials

A single material constraint is a restriction on deformation of the three-dimensional body of the form $\Gamma(\mathbf{F}) = 0$, where Γ is a given function [8, 9]. In general, Γ is subjected to the objectivity requirements and hence a single material constrain takes the form $\tilde{\Gamma}(\mathbf{C}) = 0$, where $\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2$. For thin membranes, the use of (16) this yields

$$\tilde{\Gamma}(\mathbf{C}) = \tilde{\Gamma}(\mathbf{C}, \lambda_\xi) = 0 . \tag{46}$$

Thus, (46) provides the equation for the determination of through-the-thickness stretch λ_ξ in terms of the tangential deformation tensor $\mathbf{C}$. If this equation may be solved for λ_ξ to yield $\lambda_\xi = \Lambda(\mathbf{C})$, then the constitutive relations for the membrane may be derived as described in the previous chapter.

There are four types of material constraints most often considered in continuum mechanics [8, 9]:

$$\begin{aligned}
&\tilde{\Gamma}(\mathbf{C}) \equiv \det \mathbf{C} - 1 = 0 \; - \textit{incompressibility constrain},\\
&\tilde{\Gamma}(\mathbf{C}) \equiv \operatorname{tr} \sqrt{\mathbf{C}} - 3 = 0 \; - \textit{Bell constrain},\\
&\tilde{\Gamma}(\mathbf{C}) \equiv \operatorname{tr} \mathbf{C} - 3 = 0 \; - \textit{Ericksen constrain},\\
&\tilde{\Gamma}(\mathbf{C}) \equiv \boldsymbol{e}_0 \bullet \mathbf{C}\boldsymbol{e}_0 - 1 = 0 \; - \textit{inextensibility constrain},
\end{aligned} \tag{47}$$

where $\boldsymbol{e}_0$ denotes the unit vector in the direction of inextensibility (not to be confused with the unit vector in the fibre direction of a transversely isotropic material). With the use of the results of chapter 3, each of these constraints with $\boldsymbol{e}_0$ taken to be the unit normal vector $\boldsymbol{n}_0$ to the undeformed mid-surface M, makes it possible to determine through-the-thickness stretch by the relation $\lambda_\xi = \Lambda(\mathbf{C})$. Moreover, it is easily seen that $\Lambda(\mathbf{C})$ so determined is an isotropic function of $\mathbf{C}$ and hence it depends on $\mathbf{C}$ only through the principal invariants (35).

For the incompressibility constrain, the use of (14) and (35) yields

$$\lambda_\xi = (\det \mathbf{F})^{-1} = (\det \mathbf{U})^{-1} = i_2^{-1/2} . \tag{48}$$

Substituting (48) into (36) we have

$$I_1 = i_1 + i_2^{-1}, \; I_2 = i_1 i_2^{-1} + i_2, \; I_3 = 1, \quad I_4 = i_2^{-1}, \; I_5 = i_2^{-2} . \tag{49}$$

In view of (22) and (35) for Bell constraint we have

$$\lambda_\xi = 3 - (\operatorname{tr} \mathbf{U})^{-1} = 3 - i_2^{-1/2} . \tag{50}$$

Substituting (50) in to (36) yields

$$\begin{aligned}
&I_1 = i_1 + 9 - 6 i_2^{-1/2} + i_2^{-1},\\
&I_2 = i_2 (1 + i_1 (9 - 6 i_2^{-1/2} + i_2^{-1})),\\
&I_3 = i_2 (9 - 6 i_2^{-1/2} + i_2^{-1}),\\
&I_4 = 9 - 6 i_2^{-1/2} + i_2^{-1},\\
&I_1 = (9 - 6 i_2^{-1/2} + i_2^{-1})^2 .
\end{aligned} \tag{51}$$

The Ericksen constrain yields the equation

$$\lambda_\xi^2 = 3 - \operatorname{tr} \mathbf{C} = 3 - i_2 . \tag{52}$$

Hence from (36) we then have

$$I_1 = i_1 + 3 i_2^{-1}, \; I_2 = 3 i_1 + i_2, \quad I_3 = 3, \; I_4 = 3 i_2^{-1}, \; I_5 = 9 i_2^{-2} . \tag{53}$$

A material that is inextensible in the direction of unit normal vector $\boldsymbol{n}_0(\boldsymbol{x})$ is characterised by the constraint

$$\boldsymbol{n}_0 \bullet \mathbf{C}\boldsymbol{n}_0 - 1 = \lambda_\xi^2 - 1 = 0 . \tag{54}$$

Thus $\lambda_\xi = 1$ and

$$I_1 = i_1 + 1, \; I_2 = i_2 (1 + i_1), \; I_3 = i_2, \quad I_4 = 1, \; I_5 = 1 . \tag{55}$$

The above result show that each of the constraints (47) uniquely determines the 3D invarinats I_A, $A = 1,2,3,4,5$, in terms of two-dimensional invarianat i_1 and i_2. Then, the two-dimensional response functions $\tau_0(i_1, i_2)$ and $\tau_1(i_1, i_2)$ for each type of these constraints are easily derived from the general formulae (45).

6. Closing remarks

The results of this work show that the two-dimensional constitutive equations for flexible elastic membranes may be derived from the constitutive equations of non-linear elasticity whenever any of the material constraints (47) is physically justified for the particular class of materials. Moreover, if a material is a transversely isotropic with the axis of the isotropy normal to the undeformed mid-surface of the membrane, then the two-dimensional response of the membrane is an isotropic.

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NETIESINĖS MEDŽIAGOS DARBĄ APIBŪDINANČIOS FUNKCIJOS SKERSAI IZOTROPINEI TAMPRIAI MEMBRANAI

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S a n t r a u k a

Membranai, pagamintai iš medžiagos, kurios fizinės savybės aprašomos deformacijos energijos funkcija W tūrio vienetui, deformacijos energijos funkcija Φ membranos vidurinio paviršiaus ploto vienetui gaunama integruojant funkciją W pagal membranos storį. (Nagrinėjama deformacijos energijos funkcija W neturi nustatytų apribojimų ir yra leidžiamos bet kokio dydžio deformacijos.) Funkcijos Φ tiksli išraiška yra išvesta skersai izotropinei medžiagai, kai izotropijos ašis sutampa su membranos nedeformuoto vidurinio paviršiaus normale. Parodoma, kad skersai izotropinei medžiagai gauta dvimatė membranos darbą apibūdinanti funkcija yra izotropinė ir išsamiai ištirta gautų fizinių priklausomybių struktūra. Tokios fizinės priklausomybės yra išvestos keturiems kontinuumo mechanikoje dažnai nagrinėjamų medžiagų tipams.

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