THE INTERVAL-VALUED INTUITIONISTIC FUZZY GEOMETRIC CHOQUET AGGREGATION OPERATOR BASED ON THE GENERALIZED BANZHAF INDEX AND 2-ADDITIVE MEASURE

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Abstract. Based on the operational laws on interval-valued intuitionistic fuzzy sets, the generalized Banzhaf interval-valued intuitionistic fuzzy geometric Choquet (GBIVIFGC) operator is proposed, which is also an interval-valued intuitionistic fuzzy value. It is worth pointing out that the GBIVIFGC operator can be seen as an extension of some geometric mean operators. Since the fuzzy measure is defined on the power set, it makes the problem exponentially complex. In order to overall reflect the interaction among elements and reduce the complexity of solving a fuzzy measure, we further introduce the GBIVIFGC operator w.r.t. 2-additive measures. Furthermore, if the information about weights of experts and attributes is incompletely known, the models of obtaining the optimal 2-additive measures on criteria set and expert set are given by using the introduced cross entropy measure and the Banzhaf index. Finally, an approach to pattern recognition and multi-criteria group decision making under interval-valued intuitionistic fuzzy environment is developed, respectively.

Keywords: interval-valued intuitionistic fuzzy set, generalized Banzhaf index, Choquet integral, 2-additive measure.

JEL Classification: A14, C02, C61, C71.

Introduction

Research background

Since Zadeh (1965) first proposed fuzzy set (FS) theory, it has been widely investigated and applied to a variety of fields. However, a fuzzy set only gives us the membership information of an element. Later, Atanassov (1986) proposed the concept of intuitionistic fuzzy set (IFS),

which is characterized by a degree of membership and a degree of non-membership. The sum of the membership degree and the non-membership degree of each ordered pair is less than or equal to one. The IFSs give us the possibility to model hesitation and uncertainty by using an additional degree (Wang 2009; Yager 2009). Atanassov (1986, 1999) further gave some basic operations and relations of IFSs. Xu (2007a) and Xu and Yager (2006) introduced some aggregation operators. The research results have been achieved in many different fields (Atanassov 1999; Hung, Yang 2008; Jiang *et al.* 2009; Bustince *et al.* 2007; Xu *et al.* 2008; Chen, Li 2010; Xu, Cai 2010a; De *et al.* 2001). However, in many real life situations, the exact values of the membership and non-membership would not be identified easily due to insufficiency in information availability. In 1989, Atanassov and Gargov (1989) introduced the concept of interval-valued intuitionistic fuzzy sets, which are characterized by an interval membership function and an interval non-membership function. Such a generalization further facilitates effectively representing inherent imprecision and uncertainty in the human decision making process. The application of interval-valued intuitionistic fuzzy sets for solving various decision making problems has received considerable attention recently (Atanassov 1994; Chen *et al.* 2011; Li 2010, 2011a, 2011b; Nayagam, Sivaraman 2011; Park *et al.* 2011; Xu, Chen 2007; Xu, Cai 2010a; Ye 2010; Yue 2011).

All above mentioned researches only considered situations where all elements in a set are independent, i.e., they only considered the addition of the importance of individual elements. However, in many practical situations, the elements are usually correlative (Grabisch 1995, 1996). As a powerful tool for measuring the interaction phenomenon among elements, fuzzy measures (Sugeno 1974) have been successfully used in decision problems (Labreuche, Grabisch 2003; Grabisch, Labreuche 2008; Yager 2003; Xu 2010; Tan, Chen 2010a, 2010b, 2011; Tan 2011a, 2011b; Tan *et al.* 2011; Zhang *et al.* 2011). Since the fuzzy measure is defined on the power set, it makes the problem exponentially complex. In order to deal with this issue, some special fuzzy measures were proposed, such as λ – fuzzy measure (Sugeno 1974), k-additive measure (Grabisch 1997) and p-symmetric fuzzy measure (Miranda *et al.* 2002). Corresponding to fuzzy measures, fuzzy integrals as a useful tool for aggregating fuzzy information are also successfully used in many different fields (Grabisch *et al.* 2000).

Research objectives

Although many aggregation operators have been developed under intuitionistic fuzzy environment, there are some problems that should be pointed out: (1) The existing aggregation operators based on the Choquet integral do not consider the interaction among elements globally; (2) How to obtain the fuzzy measure on a set is not studied; (3) The complexity of solving a fuzzy measure is not taken into account.. In order to deal with these problems, the rest of this paper is organized as follows:

In section 1, some basic notations and conceptions are reviewed, which will be used in the following. In section 2, we introduce an aggregation operator for IVIFSs by using the Choquet integral and the generalized Banzhaf index, which is named as the generalized Banzhaf interval- valued intuitionistic fuzzy geometric Choquet (GBIVIFGC) operator. In order to simplify the complexity of solving a fuzzy measure, we further define the GBIVIFGC operator w.r.t. 2-adttitive measure, which is simply expressed as the GABIVIFGC operator.

In section 3, we first give a new cross entropy measure for IVIFSs. Then, the models of getting the optimal 2-additive measures on criteria set and expert set are given, respectively. In section 4, an approach to pattern recognition and multi-criteria group decision making under interval-valued intuitionistic fuzzy environment is respectively developed. Meantime, the corresponding examples are given to illustrate the concrete application of the introduced methods. The conclusions are made in the last section.

1. Preliminaries

In this section, we will review some basic notations and definitions of interval-valued intuitionistic fuzzy sets, fuzzy measures and the Choquet integral, which will be used in the following.

1.1. Interval-valued intuitionistic fuzzy sets

Definition 1 (Atanassov, Gargov 1989)**.** Let X be a no empty finite set. An IVIFS A in X is expressed as:

$$A = \left\{ \left\langle x, [\mu_l(x), \mu_u(x)], [v_l(x), v_u(x)] \right\rangle \mid x \in X \right\},$$

where $[\mu_l(x), \mu_u(x)] \subseteq [0,1]$ and $[v_l(x), v_u(x)] \subseteq [0,1]$ respectively denote the interval-valued degrees of membership and non-membership of element $x \in X$ with the condition $\mu_u(x) + v_u(x) \leq 1$.

For any $x \in X$, $[\pi_l(x), \pi_u(x)] = [1 - \mu_u(x) - v_u(x), 1 - \mu_l(x) - v_l(x)]$ is the interval hesitance degree of element x to A. $[\pi_l(x), \pi_u(x)]$ is also called interval intuitionistic index. When $\pi_l(x) = \pi_u(x)$ for all $x \in X$, the IVIFS A degenerates to be an IFS. Especially, when $\pi_l(x) = \pi_u(x) = 0$ for all $x \in X$, the IVIFS A degenerates to be a fuzzy set.

In order to denote simply, any interval-valued intuitionistic fuzzy value (IVIFV) $\tilde{\alpha}$ can be expressed by $\tilde{\alpha} = ([a,b],[c,d])$, where $[a,b] \subseteq [0,1]$ and $[c,d] \subseteq [0,1]$ respectively denote the interval-valued degrees of membership and non-membership with the condition $b + d \leq 1$ (see Xu, Chen 2007). Let Ω be the set of all IVIFVs on X. Obviously, we know that $\tilde{\alpha}^+ = ([1,1],[0,0])$ and $\tilde{\alpha}^- = ([0,0],[1,1])$ are the largest and smallest IVIFVs, respectively (see Xu 2007b).

Combine the interval intuitionistic index, Xu (2008) introduced the following form to express an IVIFS $A = \left\{ \left\langle x, [\mu_l(x), \mu_u(x)], [v_l(x), v_u(x)], [\pi_l(x), \pi_u(x)] \right\rangle \mid x \in X \right\}$. Thus, any IVIFV $\tilde{\alpha} = ([a,b],[c,d])$ can be expressed by $\tilde{\alpha} = ([a,b],[c,d],[e,f])$ with $e = 1 - b - d$ and $f = 1 - c - a$. In this situation, it is clear that $\tilde{\alpha}^+ = ([1,1],[0,0],[0,0])$ and $\tilde{\alpha}^- = ([0,0],[1,1],[0,0])$ are the largest and smallest IVIFVs, respectively (see Xu 2008).

Definition 2 (Xu 2007b)**.** Let $\tilde{\alpha} = ([a_1,b_1],[c_1,d_1])$ and $\tilde{\gamma} = ([a_2,b_2],[c_2,d_2])$ be any two IVIFVs in Ω, then some operations of $\tilde{\alpha}$ and $\tilde{\gamma}$ are defined by:

1) $\tilde{\alpha} \otimes \tilde{\gamma} = ([a_1 a_2, b_1 b_2], [c_1 + c_2 - c_1 c_2, d_1 + d_2 - d_1 d_2])$,

2) $\tilde{\alpha}^\lambda = \left([a_1{}^\lambda, b_1{}^\lambda], [1 - (1 - c_1)^\lambda, 1 - (1 - d_1)^\lambda] \right), \ \lambda > 0$.

Proposition 1 (Xu 2007b). Let $\tilde{\alpha} = \big([a_1, b_1], [c_1, d_1]\big)$ and $\tilde{\gamma} = \big([a_2, b_2], [c_2, d_2]\big)$ be any two IVIFVs in Ω, then:

1) $\tilde{\alpha} \otimes \tilde{\gamma} = \tilde{\gamma} \otimes \tilde{\alpha}$,

2) $(\tilde{\alpha} \otimes \tilde{\gamma})^\lambda = \tilde{\alpha}^\lambda \otimes \tilde{\gamma}^\lambda \quad \forall \lambda > 0$,

3) $\tilde{\alpha}^{\lambda_1 + \lambda_2} = \tilde{\alpha}^{\lambda_1} \otimes \tilde{\alpha}^{\lambda_2} \quad \forall \lambda_1, \lambda_2 > 0$.

Definition 3. Let $\tilde{\alpha} = \big([a_1, b_1], [c_1, d_1]\big)$ and $\tilde{\gamma} = \big([a_2, b_2], [c_2, d_2]\big)$ be any two IVIFVs in Ω, then:

1) $\tilde{\alpha} \leq \tilde{\gamma}$ if and only if $a_1 \leq a_2, b_1 \leq b_2, c_1 \geq c_2$ and $d_1 \geq d_2$,

2) $\tilde{\alpha} = \tilde{\gamma}$ if and only if $\tilde{\alpha} \leq \tilde{\gamma}$ and $\tilde{\alpha} \geq \tilde{\gamma}$,

3) $\tilde{\alpha}^C = \big([c_1, d_1], [a_1, b_1]\big)$.

1.2. Fuzzy measures and the Choquet integral

In some real decision problems, it is not suitable to measure the importance of criteria by using additive measures because the independence of them is usually violated. In 1974, Sugeno (1974) introduced the concept of fuzzy measure, which is a powerful tool for modeling interaction among elements (Grabisch, Roubens 1999; Kojadinovic 2003, 2005) and dealing with decision problems (Grabisch 1995, 1996; Labreuche, Grabisch 2003; Grabisch, Labreuche 2008; Xu 2010; Tan, Chen 2010a, 2010b).

Definition 4 (Sugeno 1974)**.** A fuzzy measure μ on finite set N is a set function $\mu : P(N) \to [0,1]$ satisfying:

1) $\mu(\varnothing) = 0, \mu(N) = 1$,

2) $A \subseteq B$ implies $\mu(A) \leq \mu(B)$,

where $P(N)$ is the power set of N.

In the multi-criteria group decision making, $\mu(A)$ can be viewed as the importance of the criteria set or expert set A. As a result, in addition to the usual weights on criteria set or expert set taken separately, weights on any combination of criteria set or expert set are also defined.

From Definition 4, we know the fuzzy measure is defined on the power set, which makes the problem exponentially complex. Thus, it is not easy to get the fuzzy measure of each combination in a set when it is large. In order to reflect the interaction among elements and simplify the complexity of solving a fuzzy measure, some special fuzzy measures were introduced (Sugeno 1974; Grabisch 1997; Miranda *et al*. 2002).

Let $f : \{0,1\} \to \mathbb{R}$. be a pseudo-Boolean function. Grabisch (1997) pointed out that any fuzzy measure μ can be seen as a particular case of pseudo-Boolean function, and put under a multilinear polynomial in n variables:

$$\mu(A) = \sum_{T \subseteq N} \big[a_T \prod_{i \in T} y_i \big] \quad \forall A \subseteq N , \tag{1}$$

where $a_T \in \mathbb{R}, y = (y_1, y_2, ..., y_n) \in \{0,1\}^n$, and $y_i = 1$ iff $i \in A$.

The set of coefficients $a_T (T \subseteq N)$ in fact corresponds to the Möbius transform, denoted by $a_T = \sum\limits_{S \subseteq T} (-1)^{|T \setminus S|} \mu(S)$. Since the transform is inversible, μ can be recovered from a_T by

$$\mu(A) = \sum\limits_{B \subseteq A} a_B.$$

Definition 5 (Grabisch 1997). A fuzzy measure μ on N is said to be *k-additive* if its corresponding pseudo-Boolean function is a multilinear polynomial of degree k, i.e., $a_T = 0$ for all T such that $|T| > k$, and there exists at least one subset T with k elements such that $a_T \neq 0$.

When $k = 1$, the k-additive measure μ reduces to be an additive measure; When $k = n$, the k- additive μ is a fuzzy measure as usual. Especially, when $k = 2$, by Eq. (1) we get a 2-additive measure. For a 2-additive measure μ, one can easily get (Grabisch 1997), for any $S \subseteq N$ with $|S| \geq 2$:

$$\mu(S) = \sum_{i=1}^{n} a_i x_i + \sum_{\{i,j\} \subseteq N} a_{ij} x_i x_j = \sum_{i \in S} a_i + \sum_{\{i,j\} \subseteq S} a_{ij} = \sum_{\{i,j\} \subseteq S} \mu(i,j) - (|S| - 2) \sum_{i \in S} \mu(i), \quad (2)$$

where $\mu(i) = a_i, \mu(i,j) = a_i + a_j + a_{ij}$.

For a 2-additive measure, we only need $n(n+1)/2$ coefficients to determine it for a set with n elements.

Theorem 1 (Grabisch 1997). Let μ be a fuzzy measure on N, then μ is a 2-additive measure if and only if there exist coefficients $\mu(i)$ and $\mu(i,j)$ for all $i, j \in N$ that satisfy the following conditions:

1) $\mu(i) \geq 0 \quad \forall i \in N$,

2) $\sum\limits_{\{i,j\} \subseteq N} \mu(i,j) - (|N| - 2) \sum\limits_{i \in N} \mu(i) = 1$,

3) $\sum\limits_{i \subseteq S \setminus k} \left(\mu(i,k) - \mu(i) \right) \geq (|S| - 2) \mu(k) \quad \forall S \in N \ s.t. \ k \in S \ \text{and} \ |S| \geq 2$.

Corresponding to fuzzy measures, fuzzy integrals are important aggregation operators w.r.t. fuzzy measures, which are studied by many researchers (Sugeno 1974; Grabisch 1997; Miranda *et al.* 2002; Dubois, Prade 1988). One of the most important fuzzy integrals is the Choquet integral (Choquet 1953). Yager (2003) introduced the Choquet aggregation operator on fuzzy sets. Xu (2010) and Tan (2011b) studied some Choquet aggregation operators on IFSs and IVIFSs. The application of the Choquet integral has received considerable attention recently (Labreuche, Grabisch 2003; Grabisch, Labreuche 2008; Yager 2003; Xu 2010; Tan 2011a, 2011b). In 1996, Grabisch (1996) put forward the following concept of the Choquet integral on discrete sets.

Definition 6. Let f be a positive real-valued function on $X = \{x_1, x_2, ..., x_n\}$, and μ be a fuzzy measure on N. The discrete Choquet integral of f w.r.t. μ is defined by:

$$C_\mu (f(x_{(1)}), f(x_{(2)}), ..., f(x_{(n)})) = \sum_{i=1}^{n} f(x_{(i)}) (\mu(A_{(i)}) - \mu(A_{(i+1)})),$$

where $(\cdot)$ indicates a permutation on N such that $f(x_{(1)}) \leq f(x_{(2)}) \leq ... \leq f(x_{(n)})$, and $A_{(i)} = \{i, ..., n\}$ with $A_{(n+1)} = \varnothing$.

Based on the Choquet integral, Xu (2010) defined the following interval-valued intuitionistic fuzzy correlated averaging (IVIFCA) operator:

$$\text{IVIFCA}(\tilde{\alpha}_1, \tilde{\alpha}_2, ..., \tilde{\alpha}_n) = \overset{n}{\underset{i=1}{\oplus}}\, \tilde{\alpha}_{(i)}\left(\mu(A_{(i)}) - \mu(A_{(i+1)})\right),$$

and Xu (2010) and Tan (2011b) respectively presented the following interval-valued intuitionistic fuzzy geometric aggregation (IVIFGA) operator:

$$\text{IVIFGA}(\tilde{\alpha}_1, \tilde{\alpha}_2, ..., \tilde{\alpha}_n) = \overset{n}{\underset{i=1}{\otimes}}\, \tilde{\alpha}_{(i)}^{\,\mu(A_{(i)}) - \mu(A_{(i+1)})},$$

where $(\cdot)$ indicates a permutation on X such that $\tilde{\alpha}_{(1)} \leq \tilde{\alpha}_{(2)} \leq .... \leq \tilde{\alpha}_{(n)}$, and $A_{(i)} = \{i, ..., n\}$ with $A_{(n+1)} = \varnothing$, $\boldsymbol{m}$ is the fuzzy measure on index set $N = \{1, 2, ..., n\}$.

Although the IVIFCA and IVIFGA operators can indict the importance of any combination in a set, they can not overall reflect the interaction among all combinations $A_{(i)}(i = 1, 2, ..., n)$ as it only considers the interaction between subsets $A_{(i)}$ and $A_{(i+1)}(i = 1, 2, ..., n)$.

2. The generalized Banzhaf interval-valued intuitionistic fuzzy geometric Choquet operator

2.1. The general case

In order to measure the power or the strength of each coalition in a game rather than that of each player, Marichal (2000) gave an expectation index called the generalized Banzhaf index in game theory, which is expressed by:

$$\varphi_S(\mu, N) = \sum_{T \subseteq N \setminus S} \frac{1}{2^{|N| - |S|}}(\mu(S \cup T) - \mu(T)) \quad \forall S \subseteq N. \tag{3}$$

From Eq. (3), when $|S| = 1$, suppose $S = \{i\}$, then we get the Banzhaf function (Banzhaf 1965) as follows:

$$\varphi_i(\mu, N) = \sum_{T \subseteq N \setminus i} \frac{1}{2^{|N| - 1}}(\mu(i \cup T) - \mu(T)) \quad \forall i \in N. \tag{4}$$

When we apply Eq. (3) in the setting of multi-criteria decision making. $\varphi_S(\mu, N)$ can be viewed as the importance of the criteria set S. For any $S \subseteq N$, if there is no interaction between the coalition $S \subseteq N$ and any coalition in $N \setminus S$, then μ degenerates to be an additive measure. Namely, $\varphi_S(\mu, N) = \sum_{i \in S} \varphi_i(\mu, N) = \sum_{i \in S} \mu(i)$ for any $S \subseteq N$.

Theorem 1. Let $\mu : P(N) \rightarrow [0,1]$ be a fuzzy measure on N, then the generalized Banzhaf index φ given as Eq. (3) is also a fuzzy measure.

Proof. By Eq. (3), it is obvious that $\varphi_\varnothing(\mu, N) = 0$ and $\varphi_N(\mu, N) = 1$.

For any $S, K \subseteq N$ with $S \subseteq K$, when $|S| + 1 = |K|$, without loss of generality, suppose $S \cup i = K$. From Eqs (3) and (4), we get:

$$\varphi_S(\mu, N) = \sum_{T \subseteq N \backslash S} \frac{1}{2^{|N|-|S|}}(\mu(S \cup T) - \mu(T)) =$$

$$\sum_{T \subseteq N \backslash S \cup i} \frac{1}{2^{|N|-|S|}}(\mu(S \cup T) - \mu(T)) + \sum_{T \subseteq N \backslash S \cup i} \frac{1}{2^{|N|-|S|-1}}(\mu(S \cup T \cup i) - \mu(T \cup i))$$

and

$$\varphi_K(\mu, N) = \sum_{T \subseteq N \backslash K} \frac{1}{2^{|N|-|K|}}(\mu(K \cup T) - \mu(T)) =$$

$$\sum_{T \subseteq N \backslash S \cup i} \frac{1}{2^{|N|-|S|-1}}(\mu(S \cup T \cup i) - \mu(T)) \, .$$

According to the monotonicity of μ, it is clear that $\mu(T) \le \mu(T \cup i)$ and $\mu(S \cup T) \le \mu(S \cup T \cup i)$. Thus:

$$\varphi_S(\mu, N) \le \varphi_K(\mu, N) \, .$$

When $|S| + p = |K|$, without loss of generality, suppose $S \cup \{i_i, i_2, ..., i_p\} = K$ and let $S_1 = S \cup \{i_1\}$, $S_2 = S_1 \cup \{i_2\}, ..., S_p = S_{p-1} \cup \{i_p\}$. From above, we obtain:

$$\varphi_S(\mu, N) \le \varphi_{S_1}(\mu, N) \le .... \varphi_{S_p}(\mu, N) = \varphi_K(\mu, N) \, .$$

From Definition 4, we know that φ given as Eq. (3) is a fuzzy measure.

From Definition 6, we define the generalized Banzhaf geometric Choquet integral as follows:

$$C_{\varphi(\mu,N)}(f(x_{(1)}), f(x_{(2)}), ..., f(x_{(n)})) = \overset{n}{\underset{i=1}{\otimes}} f(x_{(i)})^{\varphi_{A_{(i)}}(\mu,N) - \varphi_{A_{(i+1)}}(\mu,N)} \, ,$$

where $(\cdot)$ indicates a permutation on N such that $f(x_{(1)}) \le f(x_{(2)}) \le ... \le f(x_{(n)})$, and $A_{(i)} = \{i, ..., n\}$ with $A_{(n+1)} = \varnothing$.

Definition 7. Let $\tilde{\alpha}_i = ([a_i, b_i], [c_i, d_i])$ $(i = 1, 2, ..., n)$ be a collection of IVIFVs in Ω, and μ be a fuzzy measure on N. A GBIVIFGC operator of $\tilde{\alpha}_i$ is defined by:

$$(CB) \int \tilde{\alpha} d\varphi(\mu, N) = \text{GBIVIFGC}(\tilde{\alpha}_1, \tilde{\alpha}_2, ..., \tilde{\alpha}_n) = \overset{n}{\underset{i=1}{\otimes}} (\tilde{\alpha}_{(i)})^{\varphi_{A_{(i)}}(\mu,N) - \varphi_{A_{(i+1)}}(\mu,N)} \, , \qquad (5)$$

where $(\cdot)$ indicates a permutation on N such that $\tilde{\alpha}_{(1)} \le \tilde{\alpha}_{(2)} \le ... \le \tilde{\alpha}_{(n)}$, and $A_{(i)} = \{i, ..., n\}$ with $A_{(n+1)} = \varnothing$.

Remark 1. When we restrict the domain of IVIFSs in the setting of IFSs, then Eq. (5) degenerates to be the generalized Banzhaf intuitionistic fuzzy geometric Choquet (GBIFGC) operator.

Theorem 2. Let $\tilde{\alpha}_i = \big([a_i, b_i], [c_i, d_i]\big)$ $(i = 1, 2, \ldots, n)$ be a collection of IVIFVs in Ω, and μ be a fuzzy measure on N. Then their aggregated value by using the GBIVIFGC operator is also an IVIFV, and denoted by:

$$\text{GBIVIFGC}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \left(\left[\prod_{i=1}^{n} a_{(i)}^{\varphi_{A_{(i)}}(\mu, N) - \varphi_{A_{(i+1)}}(\mu, N)}, \prod_{i=1}^{n} b_{(i)}^{\varphi_{A_{(i)}}(\mu, N) - \varphi_{A_{(i+1)}}(\mu, N)}\right],\right.$$

$$\left.\left[1 - \prod_{i=1}^{n}(1 - c_{(i)})^{\varphi_{A_{(i)}}(\mu, N) - \varphi_{A_{(i+1)}}(\mu, N)}, 1 - \prod_{i=1}^{n}(1 - d_{(i)})^{\varphi_{A_{(i)}}(\mu, N) - \varphi_{A_{(i+1)}}(\mu, N)}\right]\right), \qquad (6)$$

where $(\cdot)$ indicates a permutation on N such that $\tilde{\alpha}_{(1)} \leq \tilde{\alpha}_{(2)} \leq \ldots \leq \tilde{\alpha}_{(n)}$, and $A_{(i)} = \{i, \ldots, n\}$ with $A_{(n+1)} = \varnothing$.

Proof. The first result follows quickly from Definition 1. Below we prove Eq. (6) by using mathematical induction on n.

(i) When $n = 2$, from Theorem 1, we have:

$$\left(\tilde{\alpha}_{(1)}\right)^{\varphi_{A_{(1)}}(\mu, N) - \varphi_{A_{(2)}}(\mu, N)} = \left(\left[a_{(1)}^{\varphi_{A_{(1)}}(\mu, N) - \varphi_{A_{(2)}}(\mu, N)}, b_{(1)}^{\varphi_{A_{(1}}(\mu, N) - \varphi_{A_{(2)}}(\mu, N)}\right],\right.$$

$$\left.\left[1 - (1 - c_{(1)})^{\varphi_{A_{(1)}}(\mu, N) - \varphi_{A_{(2)}}(\mu, N)}, 1 - (1 - d_{(1)})^{\varphi_{A_{(1)}}(\mu, N) - \varphi_{A_{(2)}}(\mu, N)}\right]\right)$$

and

$$\left(\tilde{\alpha}_{(2)}\right)^{\varphi_{A_{(2)}}(\mu, N) - \varphi_{A_{(3)}}(\mu, N)} = \left(\left[a_{(2)}^{\varphi_{A_{(2)}}(\mu, N) - \varphi_{A_{(3)}}(\mu, N)}, b_{(2)}^{\varphi_{A_{(2)}}(\mu, N) - \varphi_{A_{(3)}}(\mu, N)}\right],\right.$$

$$\left.\left[1 - (1 - c_{(2)})^{\varphi_{A_{(2)}}(\mu, N) - \varphi_{A_{(3)}}(\mu, N)}, 1 - (1 - d_{(2)})^{\varphi_{A_{(2)}}(\mu, N) - \varphi_{A_{(3)}}(\mu, N)}\right]\right).$$

Thus:

$$\text{GBIVIFGC}(\tilde{\alpha}_1, \tilde{\alpha}_2) = \left(\tilde{\alpha}_{(1)}\right)^{\varphi_{A_{(1)}}(\mu, N) - \varphi_{A_{(2)}}(\mu, N)} \otimes \left(\tilde{\alpha}_{(2)}\right)^{\varphi_{A_{(2)}}(\mu, N) - \varphi_{A_{(3)}}(\mu, N)} =$$

$$\left(\left[a_{(1)}^{\varphi_{A_{(1)}}(\mu, N) - \varphi_{A_{(2)}}(\mu, N)} a_{(2)}^{\varphi_{A_{(2)}}(\mu, N) - \varphi_{A_{(3)}}(\mu, N)},\right.\right.$$

$$\left.b_{(1)}^{\varphi_{A_{(1)}}(\mu, N) - \varphi_{A_{(2)}}(\mu, N)} b_{(2)}^{\varphi_{A_{(2)}}(\mu, N) - \varphi_{A_{(3)}}(\mu, N)}\right],$$

$$\left[1 - (1 - c_{(1)})^{\varphi_{A_{(1)}}(\mu, N) - \varphi_{A_{(2)}}(\mu, N)}(1 - c_{(2)})^{\varphi_{A_{(2)}}(\mu, N) - \varphi_{A_{(3)}}(\mu, N)},\right.$$

$$\left.\left.1 - (1 - d_{(1)})^{\varphi_{A_{(1)}}(\mu, N) - \varphi_{A_{(2)}}(\mu, N)}(1 - d_{(2)})^{\varphi_{A_{(2)}}(\mu, N) - \varphi_{A_{(3)}}(\mu, N)}\right]\right) =$$

$$\left(\left[\prod_{i=1}^{2} a_{(i)}^{\varphi_{A_{(i)}}(\mu, N) - \varphi_{A_{(i+1)}}(\mu, N)}, \prod_{i=1}^{2} b_{(i)}^{\varphi_{A_{(i)}}(\mu, N) - \varphi_{A_{(i+1)}}(\mu, N)}\right],\right.$$

$$\left.\left[1 - \prod_{i=1}^{2}(1 - c_{(i)})^{\varphi_{A_{(i)}}(\mu, N) - \varphi_{A_{(i+1)}}(\mu, N)}, 1 - \prod_{i=1}^{2}(1 - d_{(i)})^{\varphi_{A_{(i)}}(\mu, N) - \varphi_{A_{(i+1)}}(\mu, N)}\right]\right).$$

(ii) Hypothesis, Eq. (6) holds for $n = k \, (k \geq 2)$), namely,

$$\text{GBIVIFGC}(\tilde{\alpha}_1, \tilde{\alpha}_2, ..., \tilde{\alpha}_k) =$$

$$\left(\left[\prod_{i=1}^{k} a_{(i)}^{\varphi_{A(i)}(\mu,N) - \varphi_{A(i+1)}(\mu,N)}, \prod_{i=1}^{k} b_{(i)}^{\varphi_{A(i)}(\mu,N) - \varphi_{A(i+1)}(\mu,N)} \right], \right.$$
$$\left. \left[1 - \prod_{i=1}^{k} (1 - c_{(i)})^{\varphi_{A(i)}(\mu,N) - \varphi_{A(i+1)}(\mu,N)}, 1 - \prod_{i=1}^{k} (1 - d_{(i)})^{\varphi_{A(i)}(\mu,N) - \varphi_{A(i+1)}(\mu,N)} \right] \right).$$

When $n = k + 1$, by the operational laws given in Definition 2, we get:

$$\text{GBIVIFGC}(\tilde{\alpha}_1, \tilde{\alpha}_2, ..., \tilde{\alpha}_{k+1}) =$$

$$\left(\left[\prod_{i=1}^{k} a_{(i)}^{\varphi_{A(i)}(\mu,N) - \varphi_{A(i+1)}(\mu,N)}, \prod_{i=1}^{k} b_{(i)}^{\varphi_{A(i)}(\mu,N) - \varphi_{A(i+1)}(\mu,N)} \right], \right.$$
$$\left. \left[1 - \prod_{i=1}^{k} (1 - c_{(i)})^{\varphi_{A(i)}(\mu,N) - \varphi_{A(i+1)}(\mu,N)}, 1 - \prod_{i=1}^{k} (1 - d_{(i)})^{\varphi_{A(i)}(\mu,N) - \varphi_{A(i+1)}(\mu,N)} \right] \right) \otimes$$

$$\left(\left[a_{(k+1)}^{\varphi_{A(k+1)}(\mu,N) - \varphi_{A(k+2)}(\mu,N)}, b_{(k+1)}^{\varphi_{A(k+1)}(\mu,N) - \varphi_{A(k+2)}(\mu,N)} \right], \right.$$
$$\left. \left[1 - (1 - c_{(k+1)})^{\varphi_{A(k+1)}(\mu,N) - \varphi_{A(k+2)}(\mu,N)}, 1 - \prod_{i=1}^{k} (1 - d_{(k+1)})^{\varphi_{A(k+1)}(\mu,N) - \varphi_{A(k+2)}(\mu,N)} \right] \right) =$$

$$\left(\left[\prod_{i=1}^{k+1} a_{(i)}^{\varphi_{A(i)}(\mu,N) - \varphi_{A(i+1)}(\mu,N)}, \prod_{i=1}^{k+1} b_{(i)}^{\varphi_{A(i)}(\mu,N) - \varphi_{A(i+1)}(\mu,N)} \right], \right.$$
$$\left. \left[1 - \prod_{i=1}^{k+1} (1 - c_{(i)})^{\varphi_{A(i)}(\mu,N) - \varphi_{A(i+1)}(\mu,N)}, 1 - \prod_{i=1}^{k+1} (1 - d_{(i)})^{\varphi_{A(i)}(\mu,N) - \varphi_{A(i+1)}(\mu,N)} \right] \right).$$

That is, for $n = k + 1$, the Eq. (6) still holds, and the result is obtained.

From Theorem 2, it is not difficult to know the GBIVIFGC operator satisfies the properties of *commutativity*, *idempotency*, *comonotonicity* and *boundary*, which are important to an aggregation operator.

2.2. The GBIVIFGC operator w.r.t. 2-additive measure

In this section, we shall research the GABIVIFGC operator, which not only globally reflects the interaction among elements, but also greatly simplifies the complexity of solving a fuzzy measure.

Theorem 3. Let μ be a 2-fuzzy measure on N, then the generalized Banzhaf index φ for μ can be expressed by:

$$\varphi_S(\mu,N) = \sum_{\{i,j\} \subseteq S} \mu(i,j) + \frac{1}{2} \sum_{i \in S, j \in N \setminus S} \left(\mu(i,j) - |S| \mu(j) \right) - \frac{|N| + |S| - 4}{2} \sum_{i \in S} \mu(i), \tag{7}$$

for any $S \subseteq N$ with $|S| \geq 2$, and when $|S| = 1$,

$$\varphi_i(\mu, N) = \frac{3 - |N|}{2}\mu(i) + \frac{1}{2}\sum_{j \in N \setminus i}\big(\mu(i,j) - \mu(j)\big), \tag{8}$$

for any $\{i\} = S \subseteq N$.

Proof. For Eq. (7): By Eqs (2) and (3), we get:

$$\varphi_S(\mu, N) = \sum_{T \subseteq N \setminus S}\frac{1}{2^{|N|-|S|}}(\mu(S \cup T) - \mu(T)) =$$

$$\sum_{T \subseteq N \setminus S}\frac{1}{2^{|N|-|S|}}\left(\left(\sum_{\{i,j\}\subseteq S \cup T}\mu(i,j) - (|S|+|T|-2)\sum_{i \in S \cup T}\mu(i)\right) - \right.$$

$$\left.\left(\sum_{\{i,j\}\subseteq T}\mu(i,j) - (|T|-2)\sum_{i \in T}\mu(i)\right)\right) =$$

$$\sum_{T \subseteq N \setminus S}\frac{1}{2^{|N|-|S|}}\left(\sum_{\{i,j\}\subseteq S}\mu(i,j) + \sum_{i \subseteq S, j \in T}\mu(i,j) - (|S|+|T|-2)\sum_{i \in S}\mu(i) - |S|\sum_{j \in T}\mu(j)\right) =$$

$$\sum_{T \subseteq N \setminus S}\frac{1}{2^{|N|-|S|}}\sum_{\{i,j\}\subseteq S}\mu(i,j) - \sum_{T \subseteq N \setminus S}\frac{|S|}{2^{|N|-|S|}}\sum_{j \in T}\mu(j) + \sum_{T \subseteq N \setminus S}\frac{1}{2^{|N|-|S|}}\sum_{i \subseteq S, j \in T}\mu(i,j) -$$

$$\sum_{T \subseteq N \setminus S}\frac{(|S|+|T|-2)}{2^{|N|-|S|}}\sum_{i \in S}\mu(i) =$$

$$\sum_{\{i,j\}\subseteq S}\frac{\displaystyle\sum_{|T|=0}^{|N|-|S|}C_{|N|-|S|}^{|T|}}{2^{|N|-|S|}}\mu(i,j) + \sum_{i \subseteq S, j \in N \setminus S}\frac{\displaystyle\sum_{|T|=0}^{|N|-|S|-1}C_{|N|-|S|-1}^{|T|}}{2^{|N|-|S|}}\mu(i,j) - \sum_{j \in N \setminus S}\frac{|S|\displaystyle\sum_{|T|=0}^{|N|-|S|-1}C_{|N|-|S|-1}^{|T|}}{2^{|N|-|S|}}\mu(j) -$$

$$\sum_{i \in S}\frac{\displaystyle\sum_{|T|=0}^{|N|-|S|}|T|C_{|N|-|S|}^{|T|} + (|S|-2)\sum_{|T|=0}^{|N|-|S|}C_{|N|-|S|}^{|T|}}{2^{|N|-|S|}}\mu(i) =$$

$$\sum_{\{i,j\}\subseteq S}\frac{2^{|N|-|S|}}{2^{|N|-|S|}}\mu(i,j) + \sum_{i \subseteq S, j \in N \setminus S}\frac{2^{|N|-|S|-1}}{2^{|N|-|S|}}\mu(i,j) - \sum_{j \in N \setminus S}\frac{|S|2^{|N|-|S|-1}}{2^{|N|-|S|}}\mu(j) -$$

$$\sum_{i \in S}\frac{\big(|N|-|S|\big)2^{|N|-|S|-1} + (|S|-2)2^{|N|-|S|}}{2^{|N|-|S|}}\mu(i) =$$

$$\sum_{\{i,j\}\subseteq S}\mu(i,j) + \sum_{i \subseteq S, j \in N \setminus S}\frac{1}{2}\mu(i,j) - \sum_{j \in N \setminus S}\frac{|S|}{2}\mu(j) - \sum_{i \in S}\frac{|N|+|S|-4}{2}\mu(i) =$$

$$\sum_{\{i,j\}\subseteq S}\mu(i,j) + \frac{1}{2}\sum_{i \in S, j \in N \setminus S}\big(\mu(i,j) - |S|\mu(j)\big) - \frac{|N|+|S|-4}{2}\sum_{i \in S}\mu(i).$$

For Eq. (8): Similar to Eq. (7), we have:

$$\varphi_i(\mu,N) = \sum_{T \subseteq N \setminus i} \frac{1}{2^{|N|-1}} (\mu(i \cup T) - \mu(T)) =$$

$$\frac{1}{2^{|N|-1}} \mu(i) + \frac{1}{2^{|N|-1}} \sum_{T \subseteq N \setminus i} \left(\sum_{j \in T} \left(\mu(i,j) - \mu(j) \right) - (|T|-1)\mu(i) \right) =$$

$$\frac{1}{2^{|N|-1}} \mu(i) + \sum_{j \in N \setminus i} \frac{\sum_{|T|=0}^{|N|-2} C_{|N|-2}^{|T|}}{2^{|N|-1}} \left(\mu(i,j) - \mu(j) \right) - \sum_{|T|=1}^{|N|-1} \frac{(|T|-1)C_{|N|-1}^{|T|}}{2^{|N|-1}} \mu(i) =$$

$$\frac{1}{2^{|N|-1}} \mu(i) + \sum_{j \in N \setminus i} \frac{2^{|N|-2}}{2^{|N|-1}} \left(\mu(i,j) - \mu(j) \right) - \frac{(|N|-1)2^{|N|-2} - 2^{|N|-1} + 1}{2^{|N|-1}} \mu(i) =$$

$$\frac{3-|N|}{2} \mu(i) + \frac{1}{2} \sum_{j \in N \setminus i} \left(\mu(i,j) - \mu(j) \right).$$

Theorem 4. Let μ be a 2-fuzzy measure on N, and φ be the generalized Banzhaf index for μ given as Eq. (3). Then,

$$\varphi_{S \cup i}(\mu,N) - \varphi_S(\mu,N) = \varphi_i(\mu,N), \tag{9}$$

for any $i \in N$ and any $S \subseteq N$ with $i \notin S$.

Proof. When $S = \varnothing$, the result obviously holds. When $|S| = 1$, without loss of generality, suppose $S = \{l\}$, and then from Theorem 3, we get:

$$\varphi_{\{i,l\}}(\mu,N) = \mu(i,l) + \frac{1}{2} \left(\sum_{k \in \{i,l\}, j \in N \setminus \{i,l\}} \mu(k,j) - 2 \sum_{j \in N \setminus \{i,l\}} \mu(j) \right) - \frac{|N|-2}{2} \left(\mu(i) + \mu(l) \right).$$

By Eq. (8), we have:

$$\varphi_{\{i,l\}}(\mu,N) - \varphi_l(\mu,N) = \left(\mu(i,l) + \frac{1}{2} \sum_{k \in \{i,l\}, j \in N \setminus \{i,l\}} \mu(k,j) - \frac{1}{2} \sum_{j \in N \setminus l} \mu(l,j) \right) -$$

$$\left(\sum_{j \in N \setminus \{i,l\}} \mu(j) - \frac{1}{2} \sum_{j \in N \setminus l} \mu(j) \right) - \frac{|N|-2}{2} \mu(i) - \frac{1}{2} \mu(l) =$$

$$\frac{1}{2} \sum_{j \in N \setminus i} \mu(i,j) - \frac{1}{2} \sum_{j \in N \setminus \{i,l\}} \mu(j) + \frac{1}{2} \mu(i) - \frac{|N|-2}{2} \mu(i) - \frac{1}{2} \mu(l) =$$

$$\frac{3-|N|}{2} \mu(i) + \frac{1}{2} \sum_{j \in N \setminus i} \left(\mu(i,j) - \mu(j) \right) = \varphi_i(\mu,N).$$

When $|S| \geq 2$, from Eq. (7) we obtain:

$$\varphi_{S \cup i}(\mu, N) - \varphi_S(\mu, N) = \sum_{j \in S} \mu(i,j) + \frac{1}{2} \left(\sum_{l \in N \setminus S \cup i} \mu(i,l) - \sum_{j \in S} \mu(i,j) \right) + \frac{|S|}{2} \mu(i) -$$

$$\frac{1}{2} \sum_{l \in N \setminus S} \mu(l) - \frac{1}{2} \sum_{j \in S} \mu(j) - \frac{|N| + |S| - 3}{2} \mu(i) =$$

$$\frac{3 - |N|}{2} \mu(i) + \frac{1}{2} \left(\sum_{j \in S} \mu(i,j) + \sum_{l \in N \setminus S \cup i} \mu(i,l) \right) - \frac{1}{2} \left(\sum_{j \in S} \mu(j) + \sum_{l \in N \setminus S} \mu(l) \right) =$$

$$\frac{3 - |N|}{2} \mu(i) + \frac{1}{2} \sum_{j \in N \setminus i} \left(\mu(i,j) - \mu(j) \right) = \varphi_i(\mu, N).$$

From Theorem 4, we know that the GABIVIFGC operator is an additive measure, namely, $\varphi_S(\mu, N) = \sum_{i \in S} \varphi_i(\mu, N)$ for any $S \subseteq N$. Then, it can be expressed by:

GABIVIFGC$(\tilde{\alpha}_1, \tilde{\alpha}_2, ..., \tilde{\alpha}_n) =$

$$\left(\left[\prod_{i=1}^{n} a_i^{\varphi_i(\mu, N)}, \prod_{i=1}^{n} b_i^{\varphi_i(\mu, N)} \right], \left[1 - \prod_{i=1}^{n} (1 - c_i)^{\varphi_i(\mu, N)}, 1 - \prod_{i=1}^{n} (1 - d_i)^{\varphi_i(\mu, N)} \right] \right), \quad (10)$$

where $\varphi_i(\mu, N)$ is the Banzhaf value of the element i given as Eq. (8).

3. A new cross entropy measure and the models of obtaining the optimal 2-additive measures

3.1. A new cross entropy measure for IVIFSs

In section 2, we have introduced two aggregation operators for IVIFSs, which are based on the Choquet integral and the generalized Banzhaf index. When the weighting vectors on expert set and criteria set are given, we can develop an approach to multi-criteria group decision making under interval-valued intuitionistic fuzzy environment by using the introduced operators. Otherwise, we first need to get their weighting vectors.

Cross entropy, as a very important method of measuring uncertain information, has been discussed by many researchers. It is used to obtain the weights of criteria (Hung, Yang 2008; Wang, Li 2011; Xia, Xu 2012; Ye 2011; Zhang, Yu 2012).

According to the entropy of a probability distribution introduced by Shannon (1948), Kullback and Leibler (1951) defined the cross entropy measure between two probability distributions $P = \{p_1, p_2, ..., p_n\}$ and $Q = \{q_1, q_2, ..., q_n\}$ as follows:

$$H(P, Q) = -\sum_{i=1}^{n} p_i \log_2 \frac{p_i}{q_i}. \quad (11)$$

Since Eq. (11) is neither symmetric nor satisfied the triangle inequality, it is not a "distance" in the formal sense.

By Eq. (11), for any two fuzzy sets $A = \{A(x_i), A(x_2), ..., A(x_n)\}$ and $B = \{B(x_1), B(x_2), ..., B(x_n)\}$ on the finite set $X = \{x_1, x_2, ..., x_n\}$, Shang and Jiang (1997) proposed a modified fuzzy cross entropy measure:

$$H_{SJ}(A,B) = \sum_{i=1}^{n} \left(A(x_i) \log_2 \frac{A(x_i)}{\frac{1}{2}\left(A(x_i) + B(x_i)\right)} + (1 - A(x_i)) \log_2 \frac{1 - A(x_i)}{1 - \frac{1}{2}\left(B(x_i) + A(x_i)\right)} \right).$$

Recently, Ye (2011) gave the cross entropy between two IVIFSs

$$A = \left\{ \left\langle x, [\mu_A^-(x), \mu_A^+(x)], [v_A^-(x), v_A^+(x)] \right\rangle \mid x \in X \right\}$$

and

$$B = \left\{ \left\langle x, [\mu_B^-(x), \mu_B^+(x)], [v_B^-(x), v_B^+(x)] \right\rangle \mid x \in X \right\}$$

as follows:

$$H_Y(A,B) = \sum_{i=1}^{n} \left(\frac{2 + \mu_A^+(x_i) - v_A^+(x_i)}{4} \log_2 \frac{2 + \mu_A^+(x_i) - v_A^+(x_i)}{\frac{1}{2}\left(4 + \mu_A^+(x_i) - v_A^+(x_i) + \mu_B^+(x_i) - v_B^+(x_i)\right)} + \right.$$

$$\left. \sum_{i=1}^{n} \frac{2 - \mu_A^+(x_i) + v_A^+(x_i)}{4} \log_2 \frac{2 - \mu_A^+(x_i) + v_A^+(x_i)}{\frac{1}{2}\left(4 - \mu_A^+(x_i) + v_A^+(x_i) - \mu_B^+(x_i) + v_B^+(x_i)\right)} \right),$$

where $\mu_T^+(x_i) = \mu_{Tl}(x_i) + \mu_{Tu}(x_i)$, $v_T^+(x_i) = v_{Tl}(x_i) + v_{Tu}(x_i)$ for $T = A, B$.

Since $H_Y(A,B)$ is not symmetric, Ye (2011) further gave a modified cross entropy measure to a symmetric form between IVIFSs A and B as follows:

$$H_Y^*(A,B) = H_Y(A,B) + H_Y(B,A). \tag{12}$$

Eq. (12) can be seen as a "distance" in the formal sense. However, there are still two problems of Eq.(12): on the one hand, for some $x_i \in X$, $H_Y(A,B)$ is not a number when $A(x_i)$ and $B(x_i)$ are both crisp values; on the other hand, it only considers the interval-valued membership and non-membership information. Considering the interval-valued hesitation information, we give another cross entropy measure between any two IVIFSs A and B on the finite set $X = \{x_1, x_2, ..., x_n\}$ as follows:

$$H(A,B) = \sum_{i=1}^{n} \frac{1}{n} \left(\frac{2 + \mu_A^+(x_i) + P_{\mu_A^+}(x_i)\pi_A^+(x_i) - v_A^+(x_i) - P_{v_A^+}(x_i)\pi_A^+(x_i)}{4} \times \right.$$

$$\log_2 \frac{3 + \mu_A^+(x_i) + P_{\mu_A^+}(x_i)\pi_A^+(x_i) - v_A^+(x_i) - P_{v_A^+}(x_i)\pi_A^+(x_i)}{\frac{1}{2}\left(6 + \mu_A^+(x_i) + P_{\mu_A^+}(x_i)\pi_A^+(x_i) - v_A^+(x_i) - P_{v_A^+}(x_i)\pi_A^+(x_i) + \mu_B^+(x_i) + P_{\mu_B^+}(x_i)\pi_B^+(x_i) - v_B^+(x_i) - P_{v_B^+}(x_i)\pi_B^+(x_i)\right)} +$$

$$\frac{2-\mu_A^+(x_i)-P_{\mu_A^+}(x_i)\pi_A^+(x_i)+v_A^+(x_i)+P_{v_A^+}(x_i)\pi_A^+(x_i)}{4}\times$$

$$\left.\log_2\frac{3-\mu_A^+(x_i)-P_{\mu_A^+}(x_i)\pi_A^+(x_i)+v_A^+(x_i)+P_{v_A^+}(x_i)\pi_A^+(x_i)}{\frac{1}{2}\left(6-\mu_A^+(x_i)-P_{\mu_A^+}(x_i)\pi_A^+(x_i)+v_A^+(x_i)+P_{v_A^+}(x_i)\pi_A^+(x_i)-\mu_B^+(x_i)-P_{\mu_B^+}(x_i)\pi_B^+(x_i)+v_B^+(x_i)+P_{v_B^+}(x_i)\pi_B^+(x_i)\right)}\right), \quad (13)$$

where $\pi_T^+(x_i)=\pi_{Tl}(x_i)+\pi_{Tu}(x_i)$, $P_{\mu_T^+}(x_i)=\dfrac{\mu_T^+(x_i)}{2}$ and $P_{v_T^+}(x_i)=\dfrac{v_T^+(x_i)}{2}$ for $T=A,B$.

Remark 2. For any IVIFS A, the possibility that the hesitation information tends to be the membership information is $P_{\mu_A^+}(x_i)=\mu_A^+(x_i)/2$, and the possibility that tends to be the non-membership information is $P_{v_A^+}(x_i)=v_A^+(x_i)/2$.

From the relationship between the hesitation information, the membership information and the non-membership information, we know Eq. (13) can be equivalently expressed by:

$$H(A,B)=\frac{1}{n}\sum_{i=1}^n\left(\frac{2+\left(\mu_A^+(x_i)-v_A^+(x_i)\right)\left(1+\pi_A^+(x_i)/2\right)}{4}\times\right.$$

$$\log_2\frac{3+\left(\mu_A^+(x_i)-v_A^+(x_i)\right)\left(1+\pi_A^+(x_i)/2\right)}{\frac{1}{2}\left(6+\left(\mu_A^+(x_i)-v_A^+(x_i)\right)\left(1+\pi_A^+(x_i)/2\right)+\left(\mu_B^+(x_i)-v_B^+(x_i)\right)\left(1+\pi_B^+(x_i)/2\right)\right)}+$$

$$\frac{2+\left(v_A^+(x_i)-\mu_A^+(x_i)\right)\left(1+\pi_A^+(x_i)/2\right)}{4}\times$$

$$\left.\log_2\frac{3+\left(v_A^+(x_i)-\mu_A^+(x_i)\right)\left(1+\pi_A^+(x_i)/2\right)}{\frac{1}{2}\left(6+\left(v_A^+(x_i)-\mu_A^+(x_i)\right)\left(1+\pi_A^+(x_i)/2\right)+\left(v_B^+(x_i)-\mu_B^+(x_i)\right)\left(1+\pi_B^+(x_i)/2\right)\right)}\right). \quad (14)$$

However, $H(A,B)$ is not symmetric, so it should be modified to a symmetric cross entropy measure as follows:

$$H^*(A,B)=H(A,B)+H(B,A). \quad (15)$$

It is not difficult to find that the cross entropy measure $H^*(A,B)$ between IVIFSs A and B satisfies the following properties:

(i) $H^*(A,B)\geq 0$;

(ii) $H^*(A,B)=0$ if and only if $A=B$;

(iii) $H^*(A,B)=H^*(A^C,B^C)$;

(iv) $H^*(A,B)=H^*(B,A)$;

(v) $H^*(A,B)\leq H^*(A,D)$, $H^*(B,D)\leq H^*(A,D)$ for any $D\in\Omega$ with $A\subseteq B\subseteq D$.

3.2. The models of obtaining the optimal 2-additive measures

Now, we consider a multi-criteria group decision making under interval-valued intuitionistic fuzzy environment. Let $A = \{a_1, a_2, ..., a_m\}$ be the set of alternatives, $C = \{c_1, c_2, ..., c_n\}$ be the set of criteria, and $E = \{e_1, e_2, ..., e_q\}$ be the set of experts. Assume that $[a_{ij}^k, b_{ij}^k]$ and $[c_{ij}^k, d_{ij}^k]$ are the interval degrees of membership and non-membership of the alternative a_i satisfies the criterion c_j given by the expert e_k, respectively, where $[a_{ij}^k, b_{ij}^k] \subseteq [0,1]$ and $[c_{ij}^k, d_{ij}^k] \subseteq [0,1]$ with $b_{ij}^k + d_{ij}^k \leq 1$. In other words, the evaluation of the alternative a_i w.r.t. the criterion c_j given by the expert e_k is an IVIFV $\tilde{\alpha}_{ij}^k = \left([a_{ij}^k, b_{ij}^k], [c_{ij}^k, d_{ij}^k] \right)$ $(i = 1, 2, ..., m; j = 1, 2, ..., n; k = 1, 2, ..., q)$. By $D^k = \left(\tilde{\alpha}_{ij}^k \right)_{m \times n}$, we denote the IVIFV matrix given by the expert e_k $(k = 1, 2, ..., q)$.

Since the experts' knowledge, skills and experiences are different, it is unreasonable to give an expert the same weight w.r.t. different criteria. Furthermore, the importance of an expert is not only determined by itself, but also influenced by other experts. Namely, it is a relative value. So it is not suitable to give the weights of the experts by using additive measures.

In order to overall reflect the interaction among experts, we shall use their Banzhaf values as their weights. According to the cross entropy principle, the following model is introduced to determine the optimal 2-additive measure on expert set E for the criterion c_j $(j = 1, 2, ..., n)$.

$$\max \sum_{k=1}^{q} \sum_{l=1}^{q} H^{\star}(P_k^j, P_l^j) \varphi_{e_k}(\mu^j, E),$$

$$s.t. \begin{cases} \sum_{e_l \subseteq S \backslash e_k} \left(\mu^j(e_k, e_l) - \mu^j(e_l) \right) \geq (|S| - 2) \mu^j(e_k), \quad \forall S \subseteq E, \forall e_k \in S, |S| \geq 2, \\ \sum_{\{e_k, e_l\} \subseteq E} \mu^j(e_k, e_l) - (|E| - 2) \sum_{e_i \in E} \mu^j(e_i) = 1, \\ \mu^j(e_k) \in W_{e_k}^j, \mu^j(e_k) \geq 0, \quad k = 1, 2, ..., q, \end{cases} \tag{16}$$

where $\varphi_{e_k}(\mu^j, E)$ is the Banzhaf value of the expert e_k $(k = 1, 2, ..., q)$, μ^j is the 2-additive measure on expert set E for the criterion c_j, P_k^j is the jth column of the IVIFV matrix D^k, and $W_{e_k}^j$ is the range of the expert e_k for the criterion c_j $(j = 1, 2, ..., n)$.

From Eq. (8), we further have:

$$\max \frac{3-q}{2} \sum_{k=1}^{q} \sum_{l=1}^{q} H^{\star}(P_k^j, P_l^j) \mu^j(e_k) + \frac{1}{2} \sum_{k=1}^{q} \sum_{l=1}^{q} H^{\star}(P_k^j, P_l^j) \left(\mu^j(e_k, e_l) - \mu^j(e_l) \right),$$

$$s.t. \begin{cases} \sum_{e_l \subseteq S \backslash e_k} \left(\mu^j(e_k, e_l) - \mu^j(e_l) \right) \geq (|S| - 2) \mu^j(e_k), \quad \forall S \subseteq E, \forall e_k \in S, |S| \geq 2, \\ \sum_{\{e_k, e_l\} \subseteq E} \mu^j(e_k, e_l) - (|E| - 2) \sum_{e_i \in E} \mu^j(e_i) = 1, \\ \mu^j(e_k) \in W_{e_k}^j, \mu^j(e_k) \geq 0, \quad k = 1, 2, ..., q. \end{cases} \tag{17}$$

After solving the model (17), we get the optimal 2-additive measure on expert set E for the criterion c_i $(j = 1, 2, ..., n)$. By Eq. (8), we get the Banzhaf values of the experts for the criterion c_i $(j = 1, 2, ..., n)$, which are used as the experts' weights.

Similar to the model for the optimal 2-additive measure on expert set, we give the following model for the optimal 2-additive measure on criteria set C.

$$\max \sum_{j=1}^{n}\sum_{l=1}^{n} H^{\star}(R_j,R_l)\varphi_{c_j}(\mu,C),$$

$$s.t.\begin{cases} \sum\limits_{c_l\subseteq S\backslash c_j}\left(\mu(c_j,c_l)-\mu(c_l)\right)\geq(|S|-2)\mu(c_j), \ \forall S\subseteq C, \forall c_j\in S, |S|\geq 2, \\ \sum\limits_{\{c_j,c_l\}\subseteq C}\mu(c_j,c_l)-(|C|-2)\sum\limits_{c_j\in C}\mu(c_j)=1, \\ \mu(c_j)\in W_{c_j},\mu(c_j)\geq 0, \ \ j=1,2,...,n, \end{cases} \tag{18}$$

where $\varphi_{c_j}(\mu,C)$ is the Banzhaf value of the criterion c_j $(j=1,2,...,n)$, μ is the 2-additive measure on criteria set C, R_j is the jth column of the comprehensive IVIFV matrix $\tilde{R}$, and W_{c_j} is the range of the criterion c_j.

From Eq. (8), we further obtain:

$$\max \frac{3-n}{2}\sum_{j=1}^{n}\sum_{l=1}^{n}H^{\star}(R_j,R_l)\mu(c_j)+\frac{1}{2}\sum_{j=1}^{n}\sum_{l=1}^{n}H^{\star}(R_j,R_l)(\mu(c_j,c_l)-\mu(c_l)),$$

$$s.t.\begin{cases} \sum\limits_{c_l\subseteq S\backslash c_j}\left(\mu(c_j,c_l)-\mu(c_l)\right)\geq(|S|-2)\mu(c_j), \ \forall S\subseteq C, \forall c_j\in S, |S|\geq 2, \\ \sum\limits_{\{c_j,c_l\}\subseteq C}\mu(c_j,c_l)-(|C|-2)\sum\limits_{c_j\in C}\mu(c_j)=1, \\ \mu(c_j)\in W_j,\mu(c_j)\geq 0, \ \ j=1,2,...,n. \end{cases} \tag{19}$$

After solving the model (19), we obtain the optimal 2-additive measure on criteria set C. By Eq. (8), we get the Banzhaf values of the criteria, which are used as the criteria's weights.

4. Approaches to pattern recognition and multi-attribute group decision making

4.1. An approach to pattern recognition

Consider a pattern recognition problem. Let $A=\{A_1,A_2,...,A_m\}$ be the set of the evaluation and selection elements, and $C=\{c_1,c_2,...,c_n\}$ be the set of features. In practical problems, due to the insufficiency in information availability, it is not easy to identify the exact values of the membership and non-membership. Here, assume that the evaluation of each element A_i w.r.t. each feature c_j $(j=1,2,...,n)$ is an IVIFV. From analysis above, we develop an approach to some pattern recognition problems under interval-valued intuitionistic fuzzy environment. The decision procedure is described as follows:

Step 1: Suppose that there exist m patterns $A=\{A_1,A_2,...,A_m\}$ and n features $C=\{c_1,c_2,...,c_n\}$. The evaluation of each pattern A_i w.r.t. the features C is an IVIFS:

$$A_i=\left\{\left\langle [b_{ij}^-,b_{ij}^+],[d_{ij}^-,d_{ij}^+]\right\rangle | j=1,2,...,n\right\} \ i=1,2,...,m.$$

Furthermore, assume that there is a sample to be recognized, which is represented by an IVIFS $\varepsilon = \left\{ \left\langle [e_j^-, e_j^+], [f_j^-, f_j^+] \right\rangle \mid j = 1, 2, ..., n \right\}$.

Step 2: Use the model (19) to get the optimal 2-additive measure on feature set C, and calculate their Banzhaf values by Eq. (8).

Step 3: Calculate the Banzhaf weighted cross entropy measure between A_i and ε ($i = 1, 2, ..., m$) using $WH^\star(A, B) = WH(A, B) + WH(B, A)$, where:

$$WH(A, B) = \sum_{i=1}^{n} \varphi_i(\mu, C) \left(\frac{2 + \left(\mu_A^+(x_i) - v_A^+(x_i) \right) \left(1 + \pi_A^+(x_i)/2 \right)}{4} \times \right.$$

$$\log_2 \frac{3 + \left(\mu_A^+(x_i) - v_A^+(x_i) \right) \left(1 + \pi_A^+(x_i)/2 \right)}{\frac{1}{2} \left(6 + \left(\mu_A^+(x_i) - v_A^+(x_i) \right) \left(1 + \pi_A^+(x_i)/2 \right) + \left(\mu_B^+(x_i) - v_B^+(x_i) \right) \left(1 + \pi_B^+(x_i)/2 \right) \right)} +$$

$$\frac{2 + \left(v_A^+(x_i) - \mu_A^+(x_i) \right) \left(1 + \pi_A^+(x_i)/2 \right)}{4} \times$$

$$\left. \log_2 \frac{3 + \left(v_A^+(x_i) - \mu_A^+(x_i) \right) \left(1 + \pi_A^+(x_i)/2 \right)}{\frac{1}{2} \left(6 + \left(v_A^+(x_i) - \mu_A^+(x_i) \right) \left(1 + \pi_A^+(x_i)/2 \right) + \left(v_B^+(x_i) - \mu_B^+(x_i) \right) \left(1 + \pi_B^+(x_i)/2 \right) \right)} \right).$$

Step 4: Select the smallest one.

Step 5: End.

Example 1. Assume that there are four kinds of minerals $A = \{A_1, A_2, A_3, A_4\}$, and a recognized sample ε, which are represented by the IVIFSs in the feature space $C = \{c_1, c_2, c_3\}$. Suppose we have the following dates.

$$A_1 = \left\{ \left\langle [0.2, 0.4], [0.4, 0.5] \right\rangle, \left\langle [0.2, 0.3], [0.4, 0.6] \right\rangle, \left\langle [0.6, 0.7], [0.2, 0.3] \right\rangle \right\},$$

$$A_2 = \left\{ \left\langle [0.2, 0.3], [0.4, 0.6] \right\rangle, \left\langle [0.3, 0.4], [0.3, 0.5] \right\rangle, \left\langle [0.3, 0.4], [0.4, 0.6] \right\rangle \right\},$$

$$A_3 = \left\{ \left\langle [0.1, 0.4], [0.5, 0.6] \right\rangle, \left\langle [0.2, 0.4], [0.4, 0.5] \right\rangle, \left\langle [0.2, 0.3], [0.4, 0.6] \right\rangle \right\},$$

$$A_4 = \left\{ \left\langle [0.2, 0.4], [0.4, 0.6] \right\rangle, \left\langle [0.3, 0.4], [0.4, 0.6] \right\rangle, \left\langle [0.4, 0.7], [0.2, 0.3] \right\rangle \right\},$$

$$\varepsilon = \left\{ \left\langle [0.2, 0.3], [0.4, 0.5] \right\rangle, \left\langle [0.2, 0.3], [0.3, 0.5] \right\rangle, \left\langle [0.3, 0.4], [0.2, 0.5] \right\rangle \right\}.$$

The importance of features is different, and their range of values are given as [0.25, 0.5], [0.3, 0.55] and [0.2, 0.35]. Our purpose is to distinguish which kind of minerals does the recognized sample ε belong to. In the following, we can utilize the proposed procedure to get the recognized sample ε belongs to which kind of minerals.

Step 1: By Eq. (15), calculate the cross entropy among A_i^1, A_i^2 and A_i^3 ($i = 1,2,3,4$), and we get:

$$H^\star(A_i^1, A_i^2) = 0.01, \ H^\star(A_i^1, A_i^3) = 0.06, \ H^\star(A_i^2, A_i^3) = 0.07,$$

where A_i^j denotes the jth element of A_i ($i = 1,2,3,4; j = 1,2,3$).

From the model (19), we get the following linear programming:

$$\max -0.035\mu(c_1) - 0.04\mu(c_2) - 0.065\mu(c_3) + 0.01\mu(c_1,c_2) + 0.06\mu(c_1,c_3) + 0.07\mu(c_2,c_3),$$

$$s.t. \begin{cases} \mu(c_1,c_2) + \mu(c_1,c_3) + \mu(c_2,c_3) - \mu(c_1) - \mu(c_2) - \mu(c_3) = 1, \\ \mu(c_1) - \mu(c_1,c_2) \le 0, \mu(c_2) - \mu(c_1,c_2) \le 0, \\ \mu(c_1) - \mu(c_1,c_3) \le 0, \mu(c_3) - \mu(c_1,c_3) \le 0, \\ \mu(c_2) - \mu(c_2,c_3) \le 0, \mu(c_3) - \mu(c_2,c_3) \le 0, \\ \mu(c_1) \in [0.25, 0.5], \\ \mu(c_2) \in [0.3, 0.55], \\ \mu(c_3) \in [0.2, 0.35]. \end{cases}$$

After solving the above model, we get the optimal 2-additve measure on feature set C as follows:

$$\mu(c_1) = 0.25, \ \mu(c_2) = 0.3, \ \mu(c_3) = 0.2, \ \mu(c_1,c_2) = 0.55, \ \mu(c_1,c_3) = 0.45, \ \mu(c_2,c_3) = 0.75.$$

By Eq. (8), we obtain the features' Banzhaf values are:

$$\varphi_1(\mu, C) = 0.25, \ \varphi_2(\mu, C) = 0.425, \ \varphi_3(\mu, C) = 0.325.$$

Step 2: Calculate the Banzhaf weighted cross entropy between A_i and ε ($i = 1,2,3,4$), we obtain:

$$WH^\star(A_1, \varepsilon) = 0.023, WH^\star(A_2, \varepsilon) = 0.006, WH^\star(A_3, \varepsilon) = 0.011, WH^\star(A_4, \varepsilon) = 0.014.$$

Step 3: From step 2, we get ε belongs to the second kind of minerals.

4.2. An approach to multi-criteria group decision making

Based on above analysis, we introduce the following approach to interval-valued intuitionistic fuzzy multi-criteria group decision making. The main decision procedure is described as follows:

Step 1: The expert e_k ($k = 1, 2, …, q$) evaluates the alternative a_i ($i = 1, 2, …, m$) w.r.t. each criterion c_j ($j = 1, 2, …, n$) by an IVIFV $\tilde{\alpha}_{ij}^k = \left([a_{ij}^k, b_{ij}^k], [c_{ij}^k, d_{ij}^k]\right)$ ($i = 1, 2, …, m; j = 1, 2, …, n;$ $k = 1, 2, …, q$), and form the IVIFV matrix $D^k = \left(\tilde{\alpha}_{ij}^k\right)_{m \times n}$ ($k = 1, 2, …, q$).

Step 2: If all criteria c_j ($j = 1, 2, …, n$) are benefit (i.e., the larger, the greater preference), then the criteria values do not need normalization. Otherwise, we normalize the decision matrix

$$D^k = \left(\tilde{\alpha}_{ij}^k\right)_{m \times n} \text{ into } Q^k = \left(\tilde{\chi}_{ij}^k\right)_{m \times n} \ (k = 1, 2, …, q), \text{ where } \tilde{\chi}_{ij}^k = \begin{cases} \tilde{\alpha}_{ij}^k \ \text{ for benefit attribute } c_j \\ \left(\tilde{\alpha}_{ij}^k\right)^C \ \text{ for cost attribute } c_j \end{cases}$$

and $\tilde{\chi}_{ij}^k = \left([e_{ij}^k, f_{ij}^k], [g_{ij}^k, h_{ij}^k]\right)$ ($i = 1, 2, …, m; j = 1, 2, …, n$) (see Xu, Cai 2010b; Xu, Hu 2010).

Step 3: Use the model (17), calculate the optimal 2-additive measure μ^j on expert set E for each criterion c_j ($j = 1, 2, …, n$).

Step 4: By Eq. (8), calculate the Banzhaf values of the experts for each criterion $c_j (j = 1, 2, …, n)$.

Step 5: For each $i = 1, 2, …, m$ and each $j = 1, 2, …, n$, use the GABIVIFGC operator:

$$\tilde{\gamma}_{ij} = GABIVIFGC(\tilde{\chi}_{ij}^1, \tilde{\chi}_{ij}^2,...,\tilde{\chi}_{ij}^q) =$$

$$\left(\left[\prod_{k=1}^{q}(e_{ij}^k)^{\varphi_{e_k}(\mu^j,E)}, \prod_{k=1}^{q}(f_{ij}^k)^{\varphi_{e_k}(\mu^j,E)} \right], \left[1 - \prod_{k=1}^{q}(1 - g_{ij}^k)^{\varphi_{e_k}(\mu^j,E)}, 1 - \prod_{k=1}^{q}(1 - g_{ij}^k)^{\varphi_{e_k}(\mu^j,E)} \right] \right),$$

to get the collective IVIFV $\tilde{\gamma}_{ij} = ([u_{ij}, v_{ij}], [x_{ij}, y_{ij}])$, and the comprehensive IVIFV matrix $R = (\tilde{\gamma}_{ij})_{m \times n}$.

Step 6: Apply the model (19), solve the optimal 2-additive measure μ on criteria set C.

Step 7: By Eq. (8), calculate the Banzhaf values of the criteria.

Step 8: Apply again the GABIVIFGC operator:

$$\tilde{\gamma}_{i} = GABIVIFGC(\tilde{\gamma}_{i1}, \tilde{\gamma}_{i2},...,\tilde{\gamma}_{in}) =$$

$$\left(\left[\prod_{j=1}^{n} u_{ij}^{\varphi_{c_j}(\mu,C)}, \prod_{j=1}^{n} v_{ij}^{\varphi_{c_j}(\mu,C)} \right], \left[1 - \prod_{j=1}^{n}(1 - x_{ij})^{\varphi_{c_j}(\mu,C)}, 1 - \prod_{j=1}^{n}(1 - y_{ij})^{\varphi_{c_j}(\mu,C)} \right] \right),$$

to get the synthetical value $\tilde{\gamma}_{i} = ([\alpha_{il}, \alpha_{iu}], [\beta_{il}, \beta_{iu}])$ of the alternative $a_i (i = 1, 2, …, m)$.

Step 9: Let

$$\tilde{\gamma}^+ = \left([\max_{i=1}^{m} \alpha_{il}, \max_{i=1}^{m} \alpha_{iu}], [\min_{i=1}^{m} \beta_{il}, \min_{i=1}^{m} \beta_{iu}] \right),$$

and

$$\tilde{\gamma}^- = \left([\min_{i=1}^{m} \alpha_{il}, \min_{i=1}^{m} \alpha_{iu}], [\max_{i=1}^{m} \beta_{il}, \max_{i=1}^{m} \beta_{iu}] \right).$$

Calculate the cross entropy measures between $H^\star(\tilde{\gamma}_i, \tilde{\gamma}^+)$ and $H^\star(\tilde{\gamma}_i, \tilde{\gamma}^-) (i = 1, 2, …, m)$.

Step 10: Let $d_i = \dfrac{H^\star(\tilde{\gamma}_i, \tilde{\gamma}^-)}{H^\star(\tilde{\gamma}_i, \tilde{\gamma}^-) + H^\star(\tilde{\gamma}_i, \tilde{\gamma}^+)}$ $(i = 1, 2, …, m)$. According to $d_i (i = 1, 2, …, m)$, select the biggest one.

Step 11: End.

Example 2. Suppose there is an investment company, which wants to invest a sum of money in the best option (adapted from Ref. Tan 2011b). There is a panel with five possible alternatives to invest the money: a_1 is a car company; a_2 is a food company; a_3 is a computer company; a_4 is an arms company; a_5 is a TV company. The investment company must take a decision according to the following four criteria: c_1 is the risk analysis; c_2 is the growth analysis; c_3 is the socio political impact analysis; c_4 is the environmental impact analysis. The five possible alternatives $a_i (i = 1,2,3,4,5)$ are to be evaluated using the interval-valued intuitionistic fuzzy information by three decision makers $e_k (k = 1,2,3)$, as listed in the following matrices:

$$D^1 = \begin{pmatrix} ([0.4,0.5],[0.3,0.4]) & ([0.4,0.6],[0.2,0.4]) & ([0.1,0.3],[0.5,0.6]) & ([0.3,0.4],[0.3,0.5]) \\ ([0.6,0.7],[0.2,0.3]) & ([0.6,0.7],[0.2,0.3]) & ([0.4,0.7],[0.1,0.2]) & ([0.5,0.6],[0.1,0.3]) \\ ([0.6,0.7],[0.1,0.2]) & ([0.5,0.6],[0.3,0.4]) & ([0.5,0.6],[0.1,0.3]) & ([0.4,0.5],[0.2,0.4]) \\ ([0.3,0.4],[0.2,0.3]) & ([0.6,0.7],[0.1,0.3]) & ([0.3,0.4],[0.1,0.2]) & ([0.3,0.7],[0.1,0.2]) \\ ([0.7,0.8],[0.1,0.2]) & ([0.3,0.5],[0.1,0.3]) & ([0.5,0.6],[0.2,0.3]) & ([0.3,0.4],[0.5,0.6]) \end{pmatrix},$$

$$D^2 = \begin{pmatrix} ([0.3,0.4],[0.4,0.5]) & ([0.5,0.6],[0.1,0.3]) & ([0.4,0.5],[0.3,0.4]) & ([0.4,0.6],[0.2,0.4]) \\ ([0.3,0.6],[0.3,0.4]) & ([0.4,0.7],[0.1,0.2]) & ([0.5,0.6],[0.2,0.3]) & ([0.6,0.7],[0.2,0.3]) \\ ([0.6,0.8],[0.1,0.2]) & ([0.5,0.6],[0.1,0.2]) & ([0.5,0.7],[0.2,0.3]) & ([0.1,0.3],[0.5,0.6]) \\ ([0.4,0.5],[0.3,0.5]) & ([0.5,0.8],[0.1,0.2]) & ([0.2,0.5],[0.3,0.4]) & ([0.4,0.7],[0.1,0.2]) \\ ([0.6,0.7],[0.2,0.3]) & ([0.6,0.7],[0.1,0.2]) & ([0.5,0.7],[0.2,0.3]) & ([0.6,0.7],[0.1,0.3]) \end{pmatrix},$$

$$D^3 = \begin{pmatrix} ([0.2,0.5],[0.3,0.4]) & ([0.4,0.5],[0.1,0.2]) & ([0.3,0.6],[0.2,0.3]) & ([0.3,0.7],[0.1,0.3]) \\ ([0.2,0.7],[0.2,0.3]) & ([0.3,0.6],[0.2,0.4]) & ([0.4,0.7],[0.1,0.2]) & ([0.5,0.8],[0.1,0.2]) \\ ([0.5,0.6],[0.3,0.4]) & ([0.7,0.8],[0.1,0.2]) & ([0.5,0.6],[0.2,0.3]) & ([0.4,0.5],[0.3,0.4]) \\ ([0.3,0.6],[0.2,0.4]) & ([0.4,0.6],[0.2,0.3]) & ([0.1,0.4],[0.3,0.6]) & ([0.3,0.7],[0.1,0.2]) \\ ([0.6,0.7],[0.1,0.3]) & ([0.5,0.6],[0.3,0.4]) & ([0.5,0.6],[0.2,0.3]) & ([0.5,0.6],[0.2,0.4]) \end{pmatrix}.$$

Assume that the interval-valued weighting vectors of the experts are respectively given as:

$$w_1= ([0.3, 0.5], [0.4, 0.5], [0.1, 0.2], [0.15, 0.3]),$$

$$w_2= ([0.2, 0.3], [0.2, 0.3], [0.3, 0.4], [0.35, 0.5]),$$

$$w_3= ([0.25, 0.4], [0.2, 0.3], [0.25, 0.3], [0.3, 0.45]),$$

and the interval-valued weighting vector of the criteria is given by $w = ([0.3, 0.4], [0.15, 0.25], [0.2, 0.25] [0.25, 0.3])$. In the following, we can utilize the proposed procedure to get the most desirable alternative(s).

Step 1: Since c_2 (the growth analysis) is benefit criterion, and c_1 (the risk analysis), c_3 (the socio– political impact analysis) and c_4 (the environmental impact analysis) are cost criteria, we need to normalize the given matrices as listed in the following.

$$Q^1 = \begin{pmatrix} ([0.3,0.4],[0.4,0.5]) & ([0.4,0.6],[0.2,0.4]) & ([0.5,0.6],[0.1,0.3]) & ([0.3,0.5],[0.3,0.4]) \\ ([0.2,0.3],[0.6,0.7]) & ([0.6,0.7],[0.2,0.3]) & ([0.1,0.2],[0.4,0.7]) & ([0.1,0.3],[0.5,0.6]) \\ ([0.1,0.2],[0.6,0.7]) & ([0.5,0.6],[0.3,0.4]) & ([0.1,0.3],[0.5,0.6]) & ([0.2,0.4],[0.4,0.5]) \\ ([0.2,0.3],[0.3,0.4]) & ([0.6,0.7],[0.1,0.3]) & ([0.1,0.2],[0.3,0.4]) & ([0.1,0.2],[0.3,0.7]) \\ ([0.1,0.2],[0.7,0.8]) & ([0.3,0.5],[0.1,0.3]) & ([0.2,0.3],[0.5,0.6]) & ([0.5,0.6],[0.3,0.4]) \end{pmatrix},$$

$$Q^2 = \begin{pmatrix} ([0.4,0.5],[0.3,0.4]) & ([0.5,0.6],[0.1,0.3]) & ([0.3,0.4],[0.4,0.5]) & ([0.2,0.4],[0.4,0.6]) \\ ([0.3,0.4],[0.3,0.6]) & ([0.4,0.7],[0.1,0.2]) & ([0.2,0.3],[0.5,0.6]) & ([0.2,0.3],[0.6,0.7]) \\ ([0.1,0.2],[0.6,0.8]) & ([0.5,0.6],[0.1,0.2]) & ([0.2,0.3],[0.5,0.7]) & ([0.5,0.6],[0.1,0.3]) \\ ([0.3,0.5],[0.4,0.5]) & ([0.5,0.8],[0.1,0.2]) & ([0.3,0.4],[0.2,0.5]) & ([0.1,0.2],[0.4,0.7]) \\ ([0.2,0.3],[0.6,0.7]) & ([0.6,0.7],[0.1,0.2]) & ([0.2,0.3],[0.5,0.7]) & ([0.1,0.3],[0.6,0.7]) \end{pmatrix},$$

$$Q^3 = \begin{pmatrix} \left([0.3,0.4],[0.2,0.5]\right) & \left([0.4,0.5],[0.1,0.2]\right) & \left([0.2,0.3],[0.3,0.6]\right) & \left([0.1,0.3],[0.3,0.7]\right) \\ \left([0.2,0.3],[0.2,0.7]\right) & \left([0.3,0.6],[0.2,0.4]\right) & \left([0.1,0.2],[0.4,0.7]\right) & \left([0.1,0.2],[0.5,0.8]\right) \\ \left([0.3,0.4],[0.5,0.6]\right) & \left([0.7,0.8],[0.1,0.2]\right) & \left([0.2,0.3],[0.5,0.6]\right) & \left([0.3,0.4],[0.4,0.5]\right) \\ \left([0.2,0.4],[0.3,0.6]\right) & \left([0.4,0.6],[0.2,0.3]\right) & \left([0.3,0.6],[0.1,0.4]\right) & \left([0.1,0.2],[0.3,0.7]\right) \\ \left([0.1,0.3],[0.6,0.7]\right) & \left([0.5,0.6],[0.3,0.4]\right) & \left([0.2,0.3],[0.5,0.6]\right) & \left([0.2,0.4],[0.5,0.6]\right) \end{pmatrix}.$$

Step 2: By Eq. (15), we get $H^\star(P_1^1, P_2^1) = 0.06$, $H^\star(P_1^1, P_3^1) = 0.062$ and $H^\star(P_2^1, P_3^1) = 0.065$. From the model (17), we have:

$$\max -0.061\mu^1(e_1) - 0.0625\mu^1(e_2) - 0.0635\mu^1(e_3) + 0.06\mu^1(e_1, e_2) + 0.062\mu^1(e_1, e_3) + 0.065\mu^1(e_2, e_3)$$

$$s.t. \begin{cases} \mu^1(e_1, e_2) + \mu^1(e_1, e_3) + \mu^1(e_2, e_3) - \mu^1(e_1) - \mu^1(e_2) - \mu^1(e_3) = 1, \\ \mu^1(e_1) - \mu^1(e_1, e_2) \le 0, \mu^1(e_2) - \mu^1(e_1, e_2) \le 0, \\ \mu^1(e_1) - \mu^1(e_1, e_3) \le 0, \mu^1(e_3) - \mu^1(e_1, e_3) \le 0, \\ \mu^1(e_2) - \mu^1(e_2, e_3) \le 0, \mu^1(e_3) - \mu^1(e_2, e_3) \le 0, \\ \mu^1(e_1) \in [0.3, 0.5], \\ \mu^1(e_2) \in [0.2, 0.3], \\ \mu^1(e_3) \in [0.25, 0.4]. \end{cases}$$

After solving the above model, we get the following optimal 2-additve measure μ^1 on E for the criterion c_1:

$$\mu^1(e_1) = 0.3, \ \mu^1(e_2) = 0.2, \ \mu^1(e_3) = 0.25, \ \mu^1(e_1, e_2) = 0.5, \ \mu^1(e_1, e_3) = 0.55, \ \mu^1(e_2, e_3) = 0.7.$$

Similarly, we have:

$$\begin{pmatrix} H^\star(P_1^2, P_2^2) \\ H^\star(P_1^2, P_3^2) \\ H^\star(P_2^2, P_3^2) \end{pmatrix} = \begin{pmatrix} 0.071 \\ 0.097 \\ 0.104 \end{pmatrix}, \begin{pmatrix} H^\star(P_1^3, P_2^3) \\ H^\star(P_1^3, P_3^3) \\ H^\star(P_2^3, P_3^3) \end{pmatrix} = \begin{pmatrix} 0.114 \\ 0.210 \\ 0.031 \end{pmatrix}, \begin{pmatrix} H^\star(P_1^4, P_2^4) \\ H^\star(P_1^4, P_3^4) \\ H^\star(P_2^4, P_3^4) \end{pmatrix} = \begin{pmatrix} 0.271 \\ 0.130 \\ 0.112 \end{pmatrix}$$

and

$$\begin{pmatrix} \mu^2(e_1) \\ \mu^2(e_2) \\ \mu^2(e_3) \\ \mu^2(e_1, e_2) \\ \mu^2(e_1, e_3) \\ \mu^2(e_2, e_3) \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.2 \\ 0.2 \\ 0.6 \\ 0.6 \\ 0.6 \end{pmatrix}, \begin{pmatrix} \mu^3(e_1) \\ \mu^3(e_2) \\ \mu^3(e_3) \\ \mu^3(e_1, e_2) \\ \mu^3(e_1, e_3) \\ \mu^3(e_2, e_3) \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.3 \\ 0.25 \\ 0.4 \\ 0.7 \\ 0.55 \end{pmatrix}, \begin{pmatrix} \mu^4(e_1) \\ \mu^4(e_2) \\ \mu^4(e_3) \\ \mu^4(e_1, e_2) \\ \mu^4(e_1, e_3) \\ \mu^4(e_2, e_3) \end{pmatrix} = \begin{pmatrix} 0.15 \\ 0.35 \\ 0.3 \\ 0.7 \\ 0.45 \\ 0.65 \end{pmatrix}.$$

Step 3: From Step 2 and Eq. (8), we get the Banzhaf values of the experts w.r.t. each criterion as listed in Table 1.

Table 1. The Banzhaf values of the experts w.r.t. each criterion

	Criterion c_1	Criterion c_2	Criterion c_3	Criterion c_4
The Banzhaf value φ_{e_1} of the expert e_1	0.3	0.4	0.275	0.25
The Banzhaf value φ_{e_2} of the expert e_2	0.325	0.3	0.3	0.45
The Banzhaf value φ_{e_3} of the expert e_3	0.375	0.3	0.425	0.3

Step 4: Use the GABIVIFGC operator, we get the following comprehensive IVIFV matrix

$$R = \begin{pmatrix} ([0.33,0.43],[0.30,0.47])([0.43,0.57],[0.14,0.31])([0.29,0.40],[0.28,0.50])([0.18,0.39],[0.35,0.59]) \\ ([0.32,0.67],[0.23,0.33])([0.43,0.67],[0.17,0.30])([0.12,0.23],[0.43,0.67])([0.14,0.27],[0.55,0.71]) \\ ([0.56,0.69],[0.18,0.28])([0.55,0.65],[0.19,0.29])([0.17,0.30],[0.50,0.63])([0.34,0.48],[0.28,0.42]) \\ ([0.33,0.50],[0.23,0.41])([0.50,0.70],[0.13,0.27])([0.22,0.39],[0.19,0.43])([0.10,0.20],[0.34,0.70]) \\ ([0.63,0.73],[0.13,0.27])([0.43,0.58],[0.17,0.30])([0.20,0.30],[0.50,0.63])([0.18,0.39],[0.51,0.61]) \end{pmatrix}.$$

Step 5: From the comprehensive IVIFV matrix $R = (\tilde{\gamma}_{ij})_{m \times n}$ and Eq. (15), we have:

$$H^\star(R_1, R_2) = 0.085, \ H^\star(R_1, R_3) = 0.557, \ H^\star(R_1, R_4) = 0.496,$$
$$H^\star(R_2, R_3) = 0.607, \ H^\star(R_2, R_4) = 0.680, \ H^\star(R_3, R_4) = 0.130.$$

By the model (19), we have:

$$\max -1.138\mu(c_1) - 1.372\mu(c_2) - 1.294\mu(c_3) - 1.306\mu(c_4) + 0.085\mu(c_1, c_2) +$$
$$0.557\mu(c_1, c_3) + 0.496\mu(c_1, c_4) + 0.607\mu(c_2, c_3) + 0.68\mu(c_2, c_4) + 0.13\mu(c_3, c_4)$$

$$s.t. \begin{cases} \mu(c_1) + \mu(c_2) - \mu(c_1, c_2) \leq 0, \mu(c_1) + \mu(c_3) - \mu(c_1, c_3) \leq 0, \\ \mu(c_1) + \mu(c_4) - \mu(c_1, c_4) \leq 0, \mu(e_2) + \mu(e_3) - \mu(e_2, e_3) \leq 0, \\ \mu(c_2) + \mu(c_4) - \mu(c_2, c_4) \leq 0, \mu(e_3) + \mu(e_4) - \mu(e_3, e_4) \leq 0, \\ \mu(c_1) + \mu(c_2) + \mu(c_3) - \mu(c_1, c_2) - \mu(c_1, c_3) \leq 0, \\ \mu(c_1) + \mu(c_2) + \mu(c_3) - \mu(c_1, c_2) - \mu(c_2, c_3) \leq 0, \\ \mu(c_1) + \mu(c_2) + \mu(c_3) - \mu(c_1, c_3) - \mu(c_2, c_3) \leq 0, \\ \mu(c_1) + \mu(c_2) + \mu(c_4) - \mu(c_1, c_2) - \mu(c_1, c_4) \leq 0, \\ \mu(c_1) + \mu(c_2) + \mu(c_4) - \mu(c_1, c_2) - \mu(c_2, c_4) \leq 0, \\ \mu(c_1) + \mu(c_2) + \mu(c_4) - \mu(c_1, c_4) - \mu(c_2, c_4) \leq 0, \\ \mu(c_1) + \mu(c_3) + \mu(c_4) - \mu(c_1, c_3) - \mu(c_1, c_4) \leq 0, \\ \mu(c_1) + \mu(c_3) + \mu(c_4) - \mu(c_1, c_3) - \mu(c_3, c_4) \leq 0, \\ \mu(c_1) + \mu(c_3) + \mu(c_4) - \mu(c_2, c_4) - \mu(c_3, c_4) \leq 0, \\ \mu(c_2) + \mu(c_3) + \mu(c_4) - \mu(c_2, c_3) - \mu(c_2, c_4) \leq 0, \\ \mu(c_2) + \mu(c_3) + \mu(c_4) - \mu(c_2, c_3) - \mu(c_3, c_4) \leq 0, \\ \mu(c_2) + \mu(c_3) + \mu(c_4) - \mu(c_2, c_4) - \mu(c_3, c_4) \leq 0, \\ \mu(c_1, c_2) + \mu(c_1, c_3) + \mu(c_1, c_4) + \mu(c_2, c_3) + \mu(c_2, c_4) \\ + \mu(c_3, c_4) - 2(\mu(c_1) + \mu(c_2) + \mu(c_3) + \mu(c_4)) = 1, \\ \mu(c_1) \in [0.3, 0.4], \mu(c_2) \in [0.15, 0.25], \mu(c_3) \in [0.2, 0.25], \mu(c_4) \in [0.25, 0.3]. \end{cases}$$

After solving e the above model, we get the following optimal 2-additve measure μ on criteria set C:

$\mu(c_1) = 0.3$, $\mu(c_2) = 0.15$, $\mu(c_3) = 0.2$, $\mu(c_4) = 0.25$, $\mu(c_1,c_2) = 0.45$, $\mu(c_1,c_3) = 0.5$, $\mu(c_1,c_4) = 0.55$, $\mu(c_2,c_3) = 0.35$, $\mu(c_2,c_4) = 0.5$, $\mu(c_3,c_4) = 0.45$.

Step 6: From Eq. (8), we obtain the criteria Banzhaf values:

$$\varphi_1(\mu,C) = 0.3, \ \varphi_2(\mu,C) = 0.2, \ \varphi_3(\mu,C) = 0.2, \ \varphi_4(\mu,C) = 03.$$

Step 7: Use again the GABIVIFGC operator, the synthetical IVIFVs of the criteria are obtained by:

$$\begin{pmatrix} \tilde{\gamma}_1 \\ \tilde{\gamma}_2 \\ \tilde{\gamma}_3 \\ \tilde{\gamma}_4 \\ \tilde{\gamma}_5 \end{pmatrix} = \begin{pmatrix} ([0.28,0.44],[0.28,0.49]) \\ ([0.22,0.41],[0.37,0.54]) \\ ([0.38,0.52],[0.29,0.41]) \\ ([0.23,0.39],[0.24,0.50]) \\ ([0.32,0.48],[0.35,0.48]) \end{pmatrix}.$$

Step 8: By $\tilde{\gamma}_i(i = 1,2,3,4,5)$, we get $\tilde{\gamma}^+ = ([0.38,0.52],[0.24,0.41])$ and $\tilde{\gamma}^- = ([0.22,0.39], [0.37,0.54])$. Calculate the cross entropy between $\tilde{\gamma}_i(i = 1,2,3,4,5)$ and $\tilde{\gamma}^+$,

$$\begin{pmatrix} H^\star(\tilde{\gamma}_1,\tilde{\gamma}^+) \\ H^\star(\tilde{\gamma}_2,\tilde{\gamma}^+) \\ H^\star(\tilde{\gamma}_3,\tilde{\gamma}^+) \\ H^\star(\tilde{\gamma}_4,\tilde{\gamma}^+) \\ H^\star(\tilde{\gamma}_5,\tilde{\gamma}^+) \end{pmatrix} = \begin{pmatrix} 0.044 \\ 0.136 \\ 0.001 \\ 0.069 \\ 0.038 \end{pmatrix},$$

and calculate the cross entropy between $\tilde{\gamma}_i$ $(i = 1,2,3,4,5)$ and $\tilde{\gamma}^-$,

$$\begin{pmatrix} H^\star(\tilde{\gamma}_1,\tilde{\gamma}^-) \\ H^\star(\tilde{\gamma}_2,\tilde{\gamma}^-) \\ H^\star(\tilde{\gamma}_3,\tilde{\gamma}^-) \\ H^\star(\tilde{\gamma}_4,\tilde{\gamma}^-) \\ H^\star(\tilde{\gamma}_5,\tilde{\gamma}^-) \end{pmatrix} = \begin{pmatrix} 0.031 \\ 0.000 \\ 0.120 \\ 0.015 \\ 0.037 \end{pmatrix}.$$

Step 9: From $H^\star(\tilde{\gamma}_i,\tilde{\gamma}^+)$ and $H^\star(\tilde{\gamma}_i,\tilde{\gamma}^-)$ $(i = 1,2,3,4,5)$, we obtain:

$$d_1 = 0.41, \ d_2 = 0, \ d_3 = 0.99, \ d_4 = 0.18, \ d_5 = 0.49.$$

Thus, $d_3 > d_5 > d_1 > d_4 > d_2$. Namely, a_3 (computer company) is the best choice.

In Step 7, if we adopt the order relationship between IVIFVs given by Xu (2007c), then we have $\tilde{\gamma}_3 > \tilde{\gamma}_5 > \tilde{\gamma}_1 > \tilde{\gamma}_4 > \tilde{\gamma}_2$, which is the same as the ranking results above.

In Step 8, if we let $\tilde{\gamma}^+ = ([1,1],[0,0])$ and $\tilde{\gamma}^- = ([0,0],[1,1])$, then we get the following cross entropy measures:

$$\begin{pmatrix} H'^\star(\tilde{\gamma}_1,\tilde{\gamma}^+) \\ H'^\star(\tilde{\gamma}_2,\tilde{\gamma}^+) \\ H'^\star(\tilde{\gamma}_3,\tilde{\gamma}^+) \\ H'^\star(\tilde{\gamma}_4,\tilde{\gamma}^+) \\ H'^\star(\tilde{\gamma}_5,\tilde{\gamma}^+) \end{pmatrix} = \begin{pmatrix} 0.095 \\ 0.112 \\ 0.065 \\ 0.108 \\ 0.044 \end{pmatrix}, \quad \begin{pmatrix} H'^\star(\tilde{\gamma}_1,\tilde{\gamma}^-) \\ H'^\star(\tilde{\gamma}_2,\tilde{\gamma}^-) \\ H'^\star(\tilde{\gamma}_3,\tilde{\gamma}^-) \\ H'^\star(\tilde{\gamma}_4,\tilde{\gamma}^-) \\ H'^\star(\tilde{\gamma}_5,\tilde{\gamma}^-) \end{pmatrix} = \begin{pmatrix} 0.075 \\ 0.061 \\ 0.107 \\ 0.063 \\ 0.118 \end{pmatrix}.$$

From $H'^\star(\tilde{\gamma}_i,\tilde{\gamma}^+)$ and $H'^\star(\tilde{\gamma}_i,\tilde{\gamma}^-)$ ($i = 1,2,3,4,5$), we have:

$$d_1 = 0.44, \ d_2 = 0.35, \ d_3 = 0.62, \ d_4 = 0.37, \ d_5 = 0.73.$$

Thus, $d_5 > d_3 > d_1 > d_4 > d_2$. Namely, a_5 (TV company) is the best choice, which is different to the ranking results obtained above.

From this example, we can find that some disadvantages for using $([1,1],[0,0])$ and $([0,0],[1,1])$ as the positive and negative ideal points, respectively, actually exist.

In this example, if the IVIFCA operator (Xu 2010) is applied to calculate the comprehensive IVIFV matrix, then it has:

$$R_1 = \begin{pmatrix} ([0.32,0.42],[0.27,0.48]) & ([0.42,0.56],[0.13,0.29]) & ([0.27,0.37],[0.29,0.53]) & ([0.19,0.39],[0.35,0.59]) \\ ([0.22,0.32],[0.30,0.68]) & ([0.46,0.66],[0.20,0.34]) & ([0.13,0.23],[0.43,0.67]) & ([0.16,0.27],[0.55,0.71]) \\ ([0.15,0.26],[0.57,0.72]) & ([0.55,0.65],[0.16,0.26]) & ([0.16,0.30],[0.50,0.63]) & ([0.35,0.48],[0.25,0.42]) \\ ([0.22,0.39],[0.32,0.51]) & ([0.51,0.69],[0.13,0.28]) & ([0.22,0.38],[0.20,0.43]) & ([0.10,0.20],[0.35,0.70]) \\ ([0.12,0.27],[0.63,0.73]) & ([0.45,0.59],[0.16,0.31]) & ([0.20,0.30],[0.50,0.63]) & ([0.20,0.39],[0.51,0.61]) \end{pmatrix}.$$

Furthermore, if the IVIFCA operator (Xu 2010) is applied to calculate the comprehensive IVIFV $\tilde{z}_i$ of the alternative a_i ($i = 1,2,3,4,5$), then it has:

$$\begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{z}_3 \\ \tilde{z}_4 \\ \tilde{z}_5 \end{pmatrix} = \begin{pmatrix} ([0.30,0.43],[0.26,0.48]) \\ ([0.23,0.36],[0.35,0.62]) \\ ([0.28,0.40],[0.30,0.47]) \\ ([0.25,0.41],[0.26,0.49]) \\ ([0.21,0.37],[0.39,0.60]) \end{pmatrix}.$$

According to the comprehensive IVIFVs $\tilde{z}_i$ ($i = 1,2,3,4,5$), we get the following cross entropy:

$$\begin{pmatrix} H^\star(\tilde{z}_1,\tilde{z}^+) \\ H^\star(\tilde{z}_2,\tilde{z}^+) \\ H^\star(\tilde{z}_3,\tilde{z}^+) \\ H^\star(\tilde{z}_4,\tilde{z}^+) \\ H^\star(\tilde{z}_5,\tilde{z}^+) \end{pmatrix} = \begin{pmatrix} 0.0000 \\ 0.0171 \\ 0.0008 \\ 0.0011 \\ 0.0194 \end{pmatrix}, \quad \begin{pmatrix} H^\star(\tilde{z}_1,\tilde{z}^-) \\ H^\star(\tilde{z}_2,\tilde{z}^-) \\ H^\star(\tilde{z}_3,\tilde{z}^-) \\ H^\star(\tilde{z}_4,\tilde{z}^-) \\ H^\star(\tilde{z}_5,\tilde{z}^-) \end{pmatrix} = \begin{pmatrix} 0.0235 \\ 0.0007 \\ 0.0179 \\ 0.0178 \\ 0.0005 \end{pmatrix},$$

where $\tilde{z}^+ = ([0.30,0.43],[0.26,0.47])$ and $\tilde{z}^- = ([0.21,0.36],[0.39,0.62])$.

From $H^*(\tilde{z}_i, \tilde{z}^+)$ and $H^*(\tilde{z}_i, \tilde{z}^-)$ $(i = 1,2,3,4,5)$, we have:

$$d_1 = 1, \ d_2 = 0.04, \ d_3 = 0.96, \ d_4 = 0.94, \ d_5 = 0.02.$$

Thus, $d_1 > d_3 > d_4 > d_2 > d_5$. Namely, a_1 (car company) is the best choice.

In this example, if the IVIFGA operator (Xu 2010; Tan 2011b) is applied to calculate the comprehensive matrix, then it has:

$$R_2 = \begin{pmatrix} ([0.32,0.42],[0.29,0.48])([0.42,0.56],[0.14,0.31])([0.25,0.35],[0.31,0.55])([0.17,0.38],[0.36,0.61]) \\ ([0.22,0.32],[0.37,0.68])([0.42,0.66],[0.20,0.34])([0.12,0.23],[0.43,0.67])([0.15,0.27],[0.56,0.72]) \\ ([0.13,0.24],[0.58,0.73])([0.53,0.64],[0.19,0.29])([0.15,0.30],[0.50,0.63])([0.31,0.46],[0.31,0.44]) \\ ([0.22,0.38],[0.32,0.53])([0.49,0.68],[0.14,0.28])([0.18,0.32],[0.22,0.43])([0.10,0.20],[0.36,0.70]) \\ ([0.11,0.27],[0.63,0.73])([0.42,0.58],[0.19,0.32])([0.20,0.30],[0.50,0.63])([0.16,0.36],[0.53,0.64]) \end{pmatrix}.$$

Furthermore, if the IVIFGA operator (Xu 2010; Tan 2011b) is applied to calculate the comprehensive IVIFV $\tilde{z}_i$ of the alternative a_i (i = 1,2, 3,4,5), then it has:

$$\begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{z}_3 \\ \tilde{z}_4 \\ \tilde{z}_5 \end{pmatrix} = \begin{pmatrix} ([0.25,0.41],[0.30,0.52]) \\ ([0.19,0.31],[0.44,0.66]) \\ ([0.21,0.37],[0.43,0.57]) \\ ([0.18,0.32],[0.29,0.55]) \\ ([0.17,0.34],[0.52,0.63]) \end{pmatrix}.$$

According to the comprehensive IVIFVs $\tilde{z}_i$ (i = 1,2,3,4,5), we get the following cross entropy:

$$\begin{pmatrix} H^*(\tilde{z}_1,\tilde{z}^+) \\ H^*(\tilde{z}_2,\tilde{z}^+) \\ H^*(\tilde{z}_3,\tilde{z}^+) \\ H^*(\tilde{z}_4,\tilde{z}^+) \\ H^*(\tilde{z}_5,\tilde{z}^+) \end{pmatrix} = \begin{pmatrix} 0.0000 \\ 0.0215 \\ 0.0075 \\ 0.0068 \\ 0.0242 \end{pmatrix}, \begin{pmatrix} H^*(\tilde{z}_1,\tilde{z}^-) \\ H^*(\tilde{z}_2,\tilde{z}^-) \\ H^*(\tilde{z}_3,\tilde{z}^-) \\ H^*(\tilde{z}_4,\tilde{z}^-) \\ H^*(\tilde{z}_5,\tilde{z}^-) \end{pmatrix} = \begin{pmatrix} 0.0387 \\ 0.0033 \\ 0.0110 \\ 0.0233 \\ 0.0002 \end{pmatrix},$$

where $\tilde{z}^+ = ([0.25,0.41],[0.29,0.52])$ and $\tilde{z}^- = ([0.17,0.31],[0.52,0.66])$.

From $H^*(\tilde{z}_i, \tilde{z}^+)$ and $H^*(\tilde{z}_i, \tilde{z}^-)$ $(i = 1,2,3,4,5)$, we have:

$$d_1 = 1, \ d_2 = 0.13, \ d_3 = 0.59, \ d_4 = 0.77, \ d_5 = 0.01.$$

Thus, $d_1 > d_4 > d_3 > d_2 > d_5$. Namely, a_1 (car company) is the best choice. With respect to the different aggregation operators, ranking results are obtained as shown in Table 2.

Table 2. Ranking results with respect to the different aggregation operators

Operator	Ranking order
The GABIVIFGC operator	$d_3 > d_5 > d_1 > d_4 > d_2$
The IVIFCA operator	$d_1 > d_3 > d_4 > d_2 > d_5$
The IVIFGA operator	$d_1 > d_4 > d_3 > d_2 > d_5$

The ranking results show that different optimal alternatives may be yielded by using different aggregation operators, and thus, the decision makers can properly select the aggregation operators according to the underlying interest and demands to each individual problem.

Conclusions

When we deal with the real decision making problems, it is unreasonable to use the additive measures because the independence among elements in a set usually violates. Based on the Choquet integral and the generalized Banzhaf index, we put forward the GBIVIFGC operator, which not only globally considers the importance of elements or their ordered positions, but also overall reflects the correlations among them or their ordered positions. In order to simplify the complexity of solving a fuzzy measure, we further introduce the GBIVIFGC operator w.r.t. 2- additive measure. Moreover, we give a new cross entropy measure for IVIFSs. When the information about weights of experts and criteria is partly known, the models of getting the optimal 2-additive measures on them are presented, respectively. Then, we develop an approach to pattern recognition and multi-criteria group decision making under interval-valued intuitionistic fuzzy environment. And one thing which is worth pointing out is that when the features, the criteria and the experts are respectively independent, we can get the corresponding methods based on additive measures.

Acknowledgments

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